Report on Beberok habilitation

The six papers A1-A6 form a series of thematically related scientific articles – the theme being Markov-type inequalities – and all appear in scientific journals. Part of the research was carried out during research visits at the University of Padua as well as at a couple of scientific institutions in South Korea. Thus these requirements for a habilitation at Jagiellonian University are satisfied. Below I will comment on the output of A1-A6 and related articles of the author.

Briefly, a compact set E in $\mathbb{R}^N$ is said to satisfy a Markov inequality if there exist positive constants r and M such that for any polynomial P and any multiindex $\alpha = (\alpha_1, ..., \alpha_N)$,

$$(1) \qquad ||D^\alpha P||_E := ||\frac{\partial^{|\alpha|} P}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}||_E \leq [M deg(P)]^{r|\alpha|} ||P||_E$$

where $|| \cdot ||_E$ is the sup-norm (L^∞-norm) on E. It suffices to verify (1) for first-order partial derivatives ($|\alpha| = \alpha_1 + \cdots + \alpha_N = 1$). Replacing $|| \cdot ||_E$ by $|| \cdot ||_{L^p(E)}$ in (1) with $1 \leq p < \infty$, i.e., using the L^p-norm on E with respect to Lebesgue measure on E, one says that E satisfies an L^p Markov inequality. The smallest r, precisely, $\mu(E) := \inf\{r : r \text{ satisfies (1) for some } M\}$ is called the Markov exponent; we write $\mu_p(E)$ for the L^p Markov exponent. A major problem in the field is to characterize those compact sets E admitting a Markov inequality. The largest known class of sets for which this property holds are those satisfying a Hölder continuity property (HCP); this says that an associated extremal plurisubharmonic function V_E (defined in $\mathbb{C}^N$) is Hölder continuous. It is not known if this condition is necessary for a set E to satisfy a Markov inequality. Since this habilitation does not address this problem, we will defer giving the definition of V_E; but we do mention two additional facts. First, for an HCP set E, the Markov exponent is the reciprocal of the Hölder exponent. Finally, a well-studied class of HCP sets are sets that are uniformly polynomially cuspoidal (UPC), which, as the name implies, allow cusps which are polynomial in nature. For a UPC set E, it was shown by M. Baran and A. Kowalska (2022) that $\mu_p(E) \geq \mu(E)$.

Essentially, this habilitation thesis deals with finding or estimation of $\mu_p(E)$ for examples and/or classes of sets E satisfying an L^p Markov inequality. In the 1980's, P. Goetgheluck had already initiated studies of both L^∞ and L^p Markov inequalities in the multivariate setting. W. Plesniak and his students (Baran, et al) have been key contributors in this area. In a 2009 *Journal of Approximation Theory* (JAT) paper, A. Kroo showed that convex bodies in $\mathbb{R}^N$ (convex, compact sets with non-empty interior) satisfy L^p Markov inequalities with exponent $r = 2$. It was known that such sets satisfy (L^∞) Markov inequalities with the same exponent $r = 2$. The techniques used/needed in the L^p case(s) are much more involved than in the L^∞ case. In the same paper, Kroo showed that so-called C^α star-like domains ($0 < \alpha \leq 1$) satisfy L^p Markov inequalities with exponent $r = 1 + 2/\alpha$ while the previously known L^∞ case exponent is $r = 1/\alpha$. These sets can have cusps. Kroo's paper did not address the question of constructing a set with a cusp (or other singularities) for which the sharp Markov exponent could be calculated. Much is known about the sharpness of L^∞ versions of Markov inequalities for various sets E; i.e., the exact value of $\mu(E)$, but the Krakow school, especially Baran, Kowalska and Beberok, have been among the leaders of the field in sharpness of L^p versions; i.e., the exact value of $\mu_p(E)$.

Paper A1 analyzes the specific region

$$\Omega = \{(x, y) \in \mathbb{R}^2 : |x| \leq y + 1,\ x^2 \geq 4y\},$$

known as the Koornwinder domain, which has cusps at $(\pm 2, 1)$. Here the author shows that for $1 \leq p < \infty$, $\mu_p(\Omega) = 4$. To show $\mu_p(\Omega) \leq 4$, he maps Ω onto a triangle via the elementary polynomial mapping $(x, y) = (u + v, uv)$. An idea of Goetgheluck is used, arising from a paper (1989, *Colloq. Math.*) in which it is shown that an L^p Markov inequality is equivalent to a type of division property. For the reverse inequality, since Ω being UPC implies $\mu_p(\Omega) \geq \mu(\Omega)$, it sufficed to construct a sequence of polynomials showing $\mu(\Omega) \geq 4$.

Paper A3 continues the study of sharp L^p Markov exponent for regions with singularities. In this work, three families of regions Ψ_k, Γ_k and Λ_k depending on a parameter k are considered. Here Λ_k has a cusp at the origin, but unlike the Koornwinder domain Ω in A1, this cusp cannot be connected to the interior of Λ_k by a straight line. Here, weighted polynomial inequalities from a 2000 *Constructive Approximation* (CA) paper of Mastroianni and Totik are used.

Paper A5 relates to a 2020 JAT paper of Kroo wherein the author proves an L^p Markov inequality for certain regions with cusps, called cuspidal piecewise graph domains. For $0 < \gamma \leq 1$, an example of a so-called *Lip*γ piecewise graph domain is the $l_\gamma-$ball

$$B_\gamma := \{x \in \mathbb{R}^N : |x_1|^\gamma + \cdots + |x_N|^\gamma \le 1\}.$$

For *Lip*γ piecewise graph domains K, Kroo proves a sharp L^p Markov exponent of $2/\gamma$ provided K can be imbedded in an affine image of B_γ with one of its vertices being on the boundary ∂K of K. One of the goals of A5 is to provide a class of domains not satisfying this imbeddability property for which the Kroo L^p Markov exponent is sharp.

Paper A4 is on a similar theme. The goal is to find the sharp L^p Markov exponent for the set

$$K = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ ax^k \le y \le f(x)\}$$

where $k \ge 2$ is an integer, $a > 0$ and f is a nonnegative, convex function on $[0,1]$ satisfying certain properties so that, in particular, K is not a cuspidal piecewise graph domain. He succeeds in giving the sharp L^p Markov exponent for this class of domains in $\mathbb{R}^2$.

The settings of papers A2 and A6 are slightly different. Instead of working on compact subsets of $\mathbb{R}^n$ as in the aforementioned four papers, in these works he considers compact subsets of certain algebraic varieties. The background for Paper A2 is the following: Plesniak (JAT, 1990) proved that if $E \subset \mathbb{R}^N$ is a $C^\infty-$determining set, i.e., $f \in C^\infty(\mathbb{R}^N)$ with $f|_E = 0$ implies $D^\alpha f|_E = 0$ for all multiindices α, then E satisfies Bernstein's property if and only if E satisfies a Markov inequality. We say E has Bernstein's property if a function $f : E \to \mathbb{R}$ is the restriction to E of a C^∞ function on all of $\mathbb{R}^N$ precisely when

$$\lim_{k\to\infty} k^r \inf\{||f - p_k||_E : p_k \in \mathcal{P}_k\} = 0$$

for each $r > 0$. Here $\mathcal{P}_k$ is the space of polynomials of degree at most k. There do exist, however, sets that satisfy Bernstein's property that are not $C^\infty-$determining; this motivates working on (subsets of) algebraic varieties. On certain algebraic varieties V in $\mathbb{R}^N$, the author defines natural subspaces $\mathcal{P}(V)$ of the full space of polynomials $\mathcal{P}$ as well as a natural notion of $C_V^\infty-$determining and proves the analogue of Plesniak's result. As indicated, this necessitates providing appropriate definitions of the associated properties in this setting; in particular, in A2, he gives his basic definition of Markov set and Markov inequality relative to a subspace of $\mathcal{P}(V)$ and provides examples (and non-examples) to illustrate its utility. Motivated by a seminal 1995 *GAFA* paper of L. Bos and P. Milman who showed that the (L^∞) Markov inequality is equivalent to a local version (and both are equivalent to a Kolomogorov-type inequality), in A6 he also needed to define the correct analogue of this local version. In A2 the main result is for a special class of algebraic hypersurfaces in $\mathbb{R}^N$ while A6 considers a slightly more general (but analogous) class of algebraic varieties, allowing the codimension to be greater than one.

While the journals in which these papers were published are not particularly strong journals, my experience is that CA and JAT are the only top journals specializing in approximation theory and hence only a small percentage of papers in this subject can be published there. Admittedly *Mathematical Inequalities and Applications* and *Bull. Malays. Math. Sci. Soc.* are not particularly strong journals; but *Dolomites Research Notes in Approximation* (DRNA) is relatively new and is trying to build up its reputation by having special issues dedicated to individuals. *Analysis Mathematica* was a highly respected joint venture between the former USSR and Hungary which fell on hard times after the fall of the Soviet Union. Like DRNA, it has been improving with the aid of special issues. I am not familiar with *Zeitschrift für Analysis und ihre Anwendungen*; it is published by the European Math. Society so I presume it is fairly reputable. I should add that these specialized articles are also very technical and require careful calculation as well as a good understanding of the literature.

Moreover, Beberok has a dozen papers on other topics; I will comment on some of them. Early in his career he worked on problems involving the Bergman kernel for certain domains (papers P1-P7). In P1, P2, P4 and P5, he obtains explicit forms of the Bergman kernel functions for certain domains by constructing a complete orthonormal basis for the space of holomorphic square-integrable functions in the domain. His calculations involve hypergeometric functions and identities relating them. Paper P3 gives an example of a domain D whose Bergman kernel has zeroes but arises as the intersection $D = D_1 \cap D_2$ of two domains D_1 and D_2 whose Bergman kernels have no zeroes (D_1, D_2 are called Lu Qi-Keng domains). His paper P7 on L^p boundedness of the Bergman projection on certain $N-$dimensional generalized Hartogs' triangles in $\mathbb{C}^N$, although appearing in a fairly obscure journal, has been cited several times. He has a few other Markov and/or Bernstein-type inequality publications aside from A1 - A6; and P9 is on a totally different subject. Regarding P12 and P13, in 1978, Morrow and Patterson constructed an explicit cubature formula for the Chebyshev weight of the second kind on the square $[-1,1]^2 \subset \mathbb{R}^2$. Without getting into definitions and details, numerical analysts then became interested in determining asymptotics of the sequence of Lebesgue constants $\{\Lambda_n\}$ associated to these Morrow-Patterson points. A group of numerical analysts in Padua conjectured that $\Lambda_n \asymp 0(n^2)$. The combined results of P12 and P13, arising from his visits to Padua, verify that this is indeed the case.

Overall, Beberok shows good competence in his area of expertise; and he clearly is unafraid of complicated calculations and estimates. The question arises as to the value of this specialized topic. My experience with the approximation theory community is that a huge majority restrict themselves to working in the real setting, mostly on (subsets of) the real line $\mathbb{R}$. There has been a growing group in the past 25-30 years working in the complex plane, combining potential theory (weighted and unweighted), complex analysis and real variable techniques. Research in real multivariate approximate theory has also expanded, especially in problems related to numerical analysis and computation. Thus within the context of approximation theory in general, the combined results of A1-A6 do offer a reasonable contribution. Furthermore, Beberok's background and research in several complex variables, together with his recent connections with the numerical analysts in Padua, bode well for the future expansion of his research directions. Indeed, this very active Padua group is unusual in the sense that they have embraced the use of pluripotential theory as a tool in their applications. Thus I would anticipate a growing collaboration with Beberok, who has clearly earned his habilitation.

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March 5, 2026