Report on the Thesis:

"Combinatorial Banach spaces and related topics: algebraic structure and self-isometries"

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Let me start admitting that writing a fair report/assessment on a Ph.D. Thesis, based on scientific arguments and impact of the research, is not always a easy task for the reviewer. Difficulties are not only related to the time required to evaluate the results, the methodology, and the new advances when compared with previous works, but with the real improvement of our current knowledge in mathematics. I tried, in any case, to write a well founded reliable report, which is not limited to number of publications and impact factors. For the sake of accessibility, I tried to reduce the number of technical definitions and technical results needed to follow this report and the conclusions therein, however, needless to say that a report on a Ph.D. Thesis will always require some technical background and expertise for a complete understanding.

The main contributions of the thesis narrate around two main themes:

✓ The study of all linear self-isometries on combinatorial Tsirelson spaces. Applications to provide a new solution to the so-called Tingley's problem on the extension of surjective linear isometries between the unit spheres of two normed spaces to surjective real linear isometries, in the case of combinatorial Tsirelson spaces.

✓ Characterize those Banach *-algebras whose multiplication is an open mapping in terms of topological stable rank, the (covering) dimension, or differential embeddings.

The doctoral candidate Ms. N. Maślany has received a strong academic formation in Poland (Jagiellonian University), the Czech Republic (Czech Academy of Sciences) and back in Poland (Jagiellonian University and University in Kielce). She has also worked during a year in a private company, where in her own words "her mathematical knowledge found new, practical applications in an industrial context".

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The dissertation is presented in the format of a collection of articles, which is also standard in many Universities (including my own University). Ms. NM has published three papers in JCR journals:

- Natalia Maślany, Isometries of combinatorial Tsirelson spaces, Proceedings of the American Mathematical Society, 151 (2023), 4475-4484.
- Natalia Maślany, On isometries and Tingley's problem for the spaces $T[\theta, S_\alpha]$, $1 \leq \alpha < \omega_1$, Studia Mathematica, 273 (2023), 285-299.
- Tomasz Kania and Natalia Maślany, Differential embeddings into algebras of topological stable rank 1, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, published online 2024, 1-25. doi:10.1017/prm.2024.108.

We have the additional guarantee of a peer-review by experts and editors in highly credible and prestigious journals. The doctoral candidate (most probably, according to the supervisor's suggestions), has carefully improved the presentation by including several chapters which help the reader to get a smooth access to the results, context, impact, and references. This has been particularly useful to this reviewer. The published papers have been fully reproduced in Chapter 5 (Publications Contents). The introduction presents the motivations for the research proposal and achievements and places the objectives in their correct time and context. There are two technical sections with the basic definitions, theory and literature overview on combinatorial Tsirelson spaces, Differential embeddings, Banach (*-)algebras, ultraproducts, and group algebras. The dissertation also includes a brief summary of the main contributions by the candidate in Chapter 4 (this is the unique section which could be, perhaps, more elaborated, despite that all results, proofs and details can be consulted in the original papers included in Chapter 5).

Let us next review the main lines. In the first main topic, we should go back to the classical Mazur-Ulan theorem, and the subsequent generalization by P. Mankiewicz in the 1970's, which affirms that every surjective isometry between two convex bodies in two real normed spaces extends to a surjective linear isometry, Tingley's problem poses the question whether the same conclusion actually holds for surjective isometries between the unit spheres, $\mathbb{S}_X$ and $\mathbb{S}_Y$, of two normed spaces X and Y, respectively. The problem remains open, even in the case of finite-dimensional spaces with dimension greater than or equal to three. T. Banack solved the case for 2-dimensional spaces in 2022 (see [13]). Intense efforts and studies have been devoted to explore Tingley's problem in concrete classes of Banach spaces, like ℓ_p-spaces, $L_p(\mu)$ spaces ($1 \leq p \leq \infty$, and their non-commutative versions, like spaces of compact linear operators $K(H)$, p-Schatten-von Neumann classes, and $B(H)$-spaces for an arbitrary complex Hilbert space H. Positive answer to Tingley's problem have been discovered for certain classes of C*-algebras, including all von Neumann algebras, further

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Jordan generalizations, and closed function algebras on locally compact Hausdorff spaces ([14, 17, 18] and [19], as well as additional references).

It is a natural methodology on problems about extension of isometries to determine first the concrete form of these maps, or the specific geometric properties of the involved spaces. The description of all linear isometries between certain classes of Banach spaces has been a challenge for functional analysts since the early stages of Functional Analysis, compare, for example, the Banach-Stone theorem and its subsequent generalizations. Monographs and surveys have been fully devoted to collect our knowledge on linear isometries in different classes of Banach spaces. I personally think that readers could welcome a citation to the books by R. Fleming and J.E. Jamison, ["Isometries on Banach spaces: Function spaces" and "Isometries on Banach spaces: Vector-valued function spaces. Vol. 2", Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 2003 and 2007] to complement the excellent survey by L. Antunes and K. Beanland [4].

The first contribution by the doctoral candidate is a kind of Banach-Stone theorem for surjective linear isometries on combinatorial Tsirelson spaces, published in a solo paper [Proc. Amer. Math Soc.'2023]. Let us note that combinatorial Tsirelson spaces are not mere sequence spaces, The pioneering Tsirelson's space was originally designed as an example of a reflexive Banach space that neither contains isomorphic copies of c_0 nor ℓ_p for any $1 \leq p < \infty$). This closed a long-standing conjecture dating back to 1974.

For the sake of brevity we shall simply observe that the results in the dissertation focus on the combinatorial Tsirelson spaces obtained via the procedure, established by Figiel and Johnson, to define a double-parameter family of Banach spaces $T[\theta, S_\alpha]$, where α is a countable ordinal and S_α is the Schreier family of order α (satisfying some natural additional conditions). According to this construction, the dual of Tsirelson's space corresponds to one of the Banach spaces in double-parameter family.

The survey by Antunes and Beanland [4], covers a result about isometries on the Tsirelson space $T[\frac{1}{n}, S_1]$, originally obtained by Beauzamy and Casazza for $n = 2$, which asserts that for each natural number $n \geq 2$, all linear isometries on $T[\frac{1}{n}, S_1]$ have very rigid form, more concretely, a linear mapping $T : T[\frac{1}{n}, S_1] \to T[\frac{1}{n}, S_1]$ is an isometry if and only if,

$$T(e_k) = \begin{cases} \varepsilon_k e_{\pi(k)}, & \text{if } 1 \leq k \leq n; \\ \varepsilon_k e_k, & \text{if } 1 \leq k \leq n, \end{cases} \quad (k \in \mathbb{N}),$$

for some $\{-1, 1\}$-valued sequence $(\varepsilon_k)_k$ and a permutation π of $\{1, \ldots, n\}$, where $(e_n)_n$ stands for the standard unit vector basis of c_{00}. It is remarked by Antunes and Beanland that "They do not have a description of the surjective isometries on $T[\theta, \mathcal{F}]$

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for many other choices of $\theta \in (0, 1)$ and regular families $\mathcal{F}$. The caseo $\mathcal{F} = S_\alpha$ for some countable ordinal $\alpha > 1$ is especially interesting".

Let us note that all the results in the thesis and papers are restricted to real linear spaces.

The above question is positively addressed by the doctoral candidate by establishing the following main results:

The first contribution concludes that all linear isometries on $T[\theta, S_1]$ with $\theta \in (0, \frac{1}{2}]$ have a very rigid form, more concretely,

Theorem 1. *Given $\theta \in (0, \frac{1}{2}]$, for each linear isometry $U : T[\theta, S_1] \to T[\theta, S_1]$, there exist a $\{-1, 1\}$-valued sequence $(\varepsilon_n)_n$ and a permutation π of the set $\{1, 2, \ldots, \lceil \theta^{-1} \rceil\}$ satisfying*

$$U(e_n) = \begin{cases} \varepsilon_n e_{\pi(n)}, & 1 \leq n \leq \lceil \theta^{-1} \rceil \\ \varepsilon_n e_n, & n > \lceil \theta^{-1} \rceil \end{cases} \quad (n \in \mathbb{N}),$$

where $(e_n)_n$ stands for the standard unit vector basis of $T[\theta, S_1]$ and $\lceil \theta^{-1} \rceil$ denotes the ceil of θ^{-1}.

Furthermore, if $\theta^{-1} \geq 2$ is an integer, each mapping defined by the previous formula is a linear isometry on $T[\theta, S_1]$, however, for $\theta^{-1} \notin \mathbb{Z}$, the reverse implication in Theorem 1 need not, in general, hold.

The conclusion is even more "drastic" or restrictive when S_1 is replaced with a countable ordinal $\alpha \geq 2$.

Theorem 2. *Let $\theta \in (0, \frac{1}{2}]$ and let $\alpha \geq 2$ be a countable ordinal. Then an operator $U : T[\theta, S_\alpha] \to T[\theta, S_\alpha]$ is an isometry if, and only if, $U(e_n) = \varepsilon_n e_n$ $(n \in \mathbb{N})$ for an appropriate $\{-1, 1\}$-valued sequence $(\varepsilon_n)_n$.*

The last two results point out the scarcity of linear isometries on on these types of combinatorial Tsirelson spaces, but at the same time show that they are all surjective.

Both theorems are published in a solo paper by the doctoral candidate in [*Proceedings of the American Mathematical Society*'2023] with 2 citations in zbMath, which partially solves the problem left open by Antunes and Beanland in [4]. This is a strong sign of scientific maturity and independence of the doctoral candidate.

The first contributions in the dissertation naturally lead to consider Tingley's problem in the setting of (real) combinatorial Tsirelson spaces (despite of the handicap that on these spaces the linear isometries have a very rigid form). The study is performed in a second solo paper by the author of this dissertation [Studia Mathematica'2023]. Prior to deal with Tingley's problem, the candidate improves her previous conclusions on linear isometries with the next real characterization of these mappings.

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Theorem 3. *Given $\theta \in (0, \frac{1}{2}]$, a linear mapping $U : T[\theta, S_1] \to T[\theta, S_1]$ is an isometry if, and only if, there exist a $\{-1, 1\}$-valued sequence $(\varepsilon_n)_n$ and a permutation π of the set $\{1, 2, \ldots, \lfloor \theta^{-1} \rfloor\}$ satisfying*

$$U(e_n) = \begin{cases} \varepsilon_n e_{\pi(n)}, & 1 \leq n \leq \lfloor \theta^{-1} \rfloor \\ \varepsilon_n e_n, & n > \lfloor \theta^{-1} \rfloor \end{cases} \quad (n \in \mathbb{N}),$$

where $(e_n)_n$ stands for the standard unit vector basis of $T[\theta, S_1]$ and $\lfloor \theta^{-1} \rfloor$ denotes the floor of θ^{-1}.

The author of this Thesis is not the first researcher who arrived to Tingley's problem in the case of combinatorial Tsirelson spaces. It must be remarked that D. Tan gave a complete positive answer to Tingley's problem in the case of $T[\frac{1}{2}, S_1]$ and in the case of the modified Tsirelson space T_M in [3]. Tan's ideas and arguments inspired N. Maślany to explore Tingley's problem for the spaces $T[\theta, S_\alpha]$ with $\theta^{-1} \geq 2$ is an integer number, and $1 \leq \alpha < \omega_1$. This is another major contribution in the Thesis under report.

Theorem 4. *Let n be a natural number ≥ 2, and let α be a countable ordinal with $1 \leq \alpha < \omega_1$. The every surjective isometry on the unit sphere of the (real) combinatorial Tsirelson space $T[\frac{1}{n}, S_\alpha]$ extends to a surjective linear isometry on $T[\frac{1}{n}, S_\alpha]$.*

I have employed some time comparing the strategies and arguments in the paper by the doctoral candidate [Studia Mathematica'2023] and those in the prequel by Tan in [3], and I agree with the author of this dissertation that the new results and advances here required more effort and significantly differ from those in [3].

The paper published in Studia Mathematica [32] has already received a couple of citations in zbMath, one of them is highly remarkable. In the very recent paper [5], R. Wang, Z. Bai, and X. Huang have a established a full characterization of all surjective isometries on the unit spheres of the complex combinatorial Tsirelson space $T[\theta, S_\alpha]$ for arbitrary $\theta \in (0, \frac{1}{2}]$ and $1 \leq \alpha < \omega_1$, and as a consequence of this characterization they provides a positive solution to Tingley's problem for the complex combinatorial Tsirelson-type space $T[\theta, S_\alpha]$. The advances concern complex combinatorial Tsirelson, and moreover, arbitrary values for $\theta \in (0, \frac{1}{2}]$ without assuming that θ^{-1} is a natural number. As I commented in previous paragraphs, problems on extension of isometries are being intensively studied in recent years, and the subject is becoming competitive. The results by the doctoral candidate have been timely published with connections in both directions from and to them.The dissertation contains an Appendix B where the arguments in [32] are refined with ideas from [5] to see how the solution to Tingley's problem in Theorem 4 works when $T[\frac{1}{n}, S_\alpha]$ is replaced with $T[\theta, S_\alpha]$, where θ^{-1} is not necessarily an integer.

I should insist on the fact that the two solo papers by the doctoral candidate are an outstanding evidence of the maturity and independence of the candidate as researcher, and the merits gathered to support her Ph.D.

All the results analysed so far narrate around isometries on combinatorial Tsirelson spaces and the unit spheres of this spaces within the frame of Tingley's problem. There is an additional research line in the thesis: The study of when the multiplication in a Banach algebra A is open, that is, determine conditions on A to guarantee that the product on A, $\mu : A \times A \to A$, $\mu(a, b) = ab$, maps every open subsets to open subsets. S. Draga and the supervisor of this thesis, T. Kania, published a pioneering work on the investigation of the openness of the multiplication in general Banach algebras in [1]. The intrinsic connections between topology and algebra become evident by the result which shows that each unital Banach algebra with open multiplication must have topological stable rank one; yet the converse fails spectacularly: finite?dimensional matrix algebras $M_n(\mathbb{C})$ have topological stable rank one but possess open multiplication only when $n = 1$ [66]. The commutative setting is very well understood from the available literature, namely, various function algebras, like algebras of continuous or bounded functions [20?27], as well as algebras of functions of bounded variation [28,29], have open multiplication (even uniformly). Komisarski's theorem in [26] reveals that for each compact space K, the following statements hold for the real Banach algebra $C_{\mathbb{R}}(K)$ of all continuous real-valued functions on K:

✓ Then multiplication in $C_{\mathbb{R}}(K)$ is uniformly open when $\dim K = 0$ (i.e., K is totally disconnected).
✓ Then multiplication in $C_{\mathbb{R}}(K)$ is weakly open but not open when $\dim K = 1$.
✓ Then multiplication in $C_{\mathbb{R}}(K)$ is not weakly open when $\dim X > 1$.

Among the new results established in this thesis, we find a complex counterpart of Komisarski's theorem [26]. The result is contained in a paper co-authored with the Ph.D. supervisor [Proceedings of the Royal Society of Edinburgh: Section A Mathematics, online'2024].

Theorem 5. *Let K be a compact space. Then the following conditions are equivalent for the algebra $C(K)$ of all continuous complex-valued functions on K:*

(i) The multiplication on $C(K)$ is open.
(ii) The multiplication on $C(K)$ is uniformly open.
(iii) The covering dimension of K is at most one.

Moreover, the multiplication of all $C(K)$-algebras is equi-uniformly open for all compact spaces K of dimension at most one.

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The proof of this results is particularly meritful, thought technical which combines several tools and a result from an unpublished work by E. Behrends. It seems to me the kind of result that should appear in books.

It is now necessary to recall some terminology. Let A and B denote two unital Banach algebras. It is said that A admits norm-controlled inversion into B if A is inverse-closed in B and there exist a unital continuous injective homomorphism $\iota : A \to B$ and a function $h : (0,\infty)^2 \to (0,\infty)$ satisfying

$$\|a^{-1}\|_{A} \leq h\left(\|a\|_{A}, \|\iota(a^{-1})\|_{B}\right),$$

for every element $a \in A$ which is invertible in B (I think the readers could appreciate if the full definition is given in these terms, the one in the thesis is a bit ambiguous, although it seems to be one of the standard styles). The algebra A is called a differential subalgebra of B whenever there exists $D > 0$ and a unital continuous injective homomorphism $\iota : A \to B$ satisfying

$$\|ab\|_A \leq D(\|a\|_A \|\iota(b)\|_B + \|\iota(a)\|_B \|b\|_A).$$

When A and B are Banach *-algebras, it is additionally required that ι is *-preserving. Differential subalgebras (especially of C*-algebras) have been extensively studied since 1990's by authors, like Kissin and Shulman [43, Proc. Edinb. Math. Soc.'1994], Gröchenig and Klotz [44-45, London Math. Soc.'2013, Math. Nachr.'2014], and more recently Samei and Shepelska [46, J. Fourier Anal. Appl.'2019]. Among the results due to Gröchenig and Klotz, we note that all differential *-subalgebras of C*-algebras have norm-controlled inversion. Consequently, differential *-subalgebras of C*-algebras are symmetric (i.e., the spectrum of each positive elements is nonnegative).

A dual Banach algebra is a Banach algebra A for which there exists Banach space E such $A = E^*$ and the multiplication of A is separately $\sigma(A, E)$-continuous. For example, all von Neumann algebras and all Banach algebras that are reflexive as Banach spaces, or biduals of Arens-regular Banach algebras are dual Banach algebras [47, Section 5]. Suppose that $A = E^*$ is a dual Banach algebra and $\iota : A \to C(K)$ is an injective homomorphism for some compact space K. It is said that A shares with X densely many points whenever there exists a dense set $Q \subset K$ such that $\iota^*(\delta_x) \in E$ for all $x \in Q$, where $\delta_x \in C(K)^*$ is the Dirac delta evaluation functional at $x \in K$.

One of the new contributions in the paper by Kania and Maślany unifies various approaches to openness of multiplication.

Theorem 6. *Let A be a unital Banach *-algebra such that there exists an injective *-homomorphism $\iota : A \to C(K)$ for some compact space K such that A has norm-controlled inversion in $C(K)$. Suppose that A is in one of the following cases:*

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(•) $A = C(K)$,
(•) $A = E^*$ *is a dual Banach algebra that shares with* K *densely many points.*

Then multiplication in A *is open at all pairs* (a, b) *of jointly non-degenerate elements of* A *(i.e.* $|a|^2 + |b|^2 = a^*a + b^*b$ *is invertible). Suppose additionally that the* **-homomorphism* ι *has dense range in* $C(K)$*. Then if* A *has open multiplication, the maximal ideal space of* A *is of dimension at most* 1.

The conclusion for $A = C(K)$ is actually an extension of a result by E. Behrends in the case of real $C(K)$-spaces [67]. Among the consequences of the previous theorem we find the following:

Theorem 7. *Suppose that* A *is an Arens-regular Banach* **-algebra that is densely embedded as a differential subalgebra of* $C(K)$ *for some compact space* K*. Then* A^{**} *has open multiplication at all pairs of jointly non-degenerate elements.*

An additional lines explored by the doctoral candidate and her advisor is the openness of the convolution multiplication on $\ell_1(\mathbb{Z})$. It should be noted that Draga and the supervisor of this thesis established in 2018 (see [1]) that $\ell_1(\mathbb{Z})$ does not have uniformly open convolution. However, the problem of determining whether the convolution product is open or not had remained open. In this dissertation, Kania and Maślany strengthen this result with the following conclusion.

Theorem 8. *Let* G *be an Abelian group of unbounded exponent (that is,* $\sup_{g\in G} o(g) = \infty$*, where* $o(g)$ *denotes the rank of an element* $g \in G$*). Then the convolution in* $\ell_1(G)$ *is not uniformly open.*

After surveying the most important contributions in the Thesis under report, it seems appropriate to highlight the timely, impact, and the future research problems which naturally raise after the work of the candidate. The dissertation also includes a section named "Future works" devoted to pose several open goals.

Needless to say that Tingley's problem remains open for general Banach spaces with dimension ≥ 3, while we finally discover a proof or a counterexample for this problem, and specially if a counterexample finally appears, it sounds interesting to explore Tingley's problem for some other concrete spaces. A recent contribution by Fakhoury (see [15, J. Funct. Anal.'2025]), deeply linked to the results in this Thesis, describes all surjective isometries of the unit sphere of real Schreier spaces of all orders and their p-convexifications, for $1 < p < \infty$, and provides a positive answer to Tingley's problem for those spaces. Another result by Golbaharan and Amiri (see [30, J. Math. Anl. Appl.'2025]) shows how the space $T[\theta, A_n]$ possesses a standard isometry group for all $0 < \theta < 1$ and $n \in \mathbb{N}$, where A_n denotes the set of all $F \subset \mathbb{N}$ with cardinality less than or equal to n. So, Tingley's problem remains timely for different classes of mixed Tsirelson spaces. Related questions are posed

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in the context of the free Banach lattice over the combinatorial Tsirelson space, like the following:

($Q1$) Can lattice isometries be characterised in the free Banach lattice over the combinatorial Tsirelson space?

($Q2$) Does the free Banach lattice over the combinatorial Tsirelson space contain an isometric or lattice isomorphic copy of ℓ_p, for $1 \leq p < \infty$?

Concerning the problem of determining when the product of a unital Banach algebra is an open mapping, the paper by Kania and Maślany in the Proceedings of the Royal Society of Edinburgh, contains several interesting challenges at the end. For example, What are further examples of (dual) Banach algebras that are approximable by jointly non-degenerate elements? What about algebras of Lipschitz functions on zero-dimensional compact spaces? In the case of convolution algebras, can the group algebra of a group with bounded exponent have (uniformly) open convolution? Is there an infinite group G for which $\ell_1(G)$ has open convolution?

This research line remains active, the doctoral candidate and her supervisor are currently working on a couple of projects about the Continuity of Convolution in Lipschitz-free Banach Spaces over Metric Groups, and on the problem of finding conditions under which a Banach algebra inherits open multiplication from its components in a short exact sequence. These are natural question to explore during a postdoc period.

Based on all the previous conclusions and thoughts, my general *assessment on this Thesis is very positive.* Ms. Maślany has done an outstanding job proving her capacity to conduct a research with solo contributions to open problems and collaborations; the final outcome *deserves, without any doubt, a top grading*, which is my final recommendation.

The whole dissertation is extremely well written, actually friendly accessible to readers and easy to follow, despite of the technicalities of some notions and involved structures. Anyway, I found some minor weaknesses and typos, which are commented below as suggestions.

Acknowledgements, line 3. I think it should be "...wing was not an easy task–".

I personally think that the chosen format for the final Bibliography is a bit misleading, it is clearly not alphabetical, nor chronological, or in order of appearance in the dissertation. This makes that checking references becomes a bit uncomfortable to the reader. (The same applies to the references in the first paper published by the doctoral candidate in the Proc. Amer. Math Soc.)

Page 4, line 24. The phrase “Tingley’s problem for complex isometries admits a known counterexample and is therefore not addressed here” sounds a bit ambiguous. Well, I perfectly understand what the author wants to affirms, however it could be misunderstood. Tingley’s problem asks whether every surjective isometry between the unit spheres of two (real or complex) normed spaces admits an extension to a surjective real linear isometry between the spaces. The conclusion is the same we can find in the Mazur–Ulam theorem and both results find certain obstacles by the existence of real linear surjections between complex Banach spaces which are not complex linear nor conjugate linear. As long as I know there is no a single counter-example to Tingley’s problem.

Page 10, line 18. “When A and B are unital Banach algebras...”

Page 11, line -13. “... non-negative (i.e. $\sigma_{\mathbf{A}}(\mathbf{a}^*\mathbf{a}) \subseteq [0,\infty)$ for all $a \in A$, which **implies** that for any $a \in A$...”

Page 15, line 10. I suggest to replace “Banach spaces [53].” with “Banach spaces (see [53, §2.e]).”

Granada, November 18, 2025

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