Extremal problems on edge colorings of graphs
Abstract

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Edge colorings of graphs and extremal graph theory are two well-known areas of graph theory with long history. The vital results in the former one are due Shannon and Vizing who studied the chromatic index of a graph. One of the most important problems in the latter area is determining the Turán number of a graph F, i.e. the maximum number of edges in a graph which does not have F as a subgraph. The notable results are due Mantel, Turán and Erdős and Stone for F being a triangle, a complete graph and a nonbipartite graph, respectively. The aim of this dissertation is to present three problems originating from the intersection of the above areas, i.e. the extremal problems related to edge colorings. Despite each of them appears in a slightly different setting, they all are focused on investigating how many colored edges can appear so that a graph satisfies a desired property.

The first problem is concentrated on maximal edge colorings, which are the proper edge colorings of a graph G of order n with $\chi'(K_n)$ colors such that adding any edge to G in any color makes it improper. Meszka and Tyniec-Motyka partially solved the problem of determining for any number of vertices and edges if there exits a graph with maximum edge coloring. In this part of the thesis we complete the solution of this problem.

The next two problems are related to colored variants of Turán problems. For a fixed forbidden graph F we consider $c \geq 1$ graphs on a common set of vertices, thinking of each graph as edges in a distinct color, and we aim to determine the maximum number of edges in each color such that a copy of F having edges from different graphs does not appear. The second part of the dissertation studies the above problem when F is path on 3 edges. Erdős and Gallai proved the asymptotically optimal bound for the Turán number of a path of a fixed length. We present a colored version of their theorem in the case of a path on three edges and any number of colors, and prove the asymptotically optimal bound.

The last part of the dissertation is focused on graphs without rainbow triangles. Aharoni et al. analyzed a colored version of a Turán problem in the undirected setting when the forbidden graph is a triangle. We present the solution of an analogous problem for directed graphs without rainbow triangles, either directed or transitive, for any number of colors. Additionally, an analogous problem in the setting of oriented graphs is solved.

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