Manhattan, May 31, 2025

## Referee report on the dissertation
## "Very Symmetric hyper-Kähler fourfolds"
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The dissertation contributes to the theory of hyperkähler manifolds from the viewpoint of classical complex and algebraic geometry. The main results are the complete classification of maximal holomorphic finite group actions on fourfolds deformation equivalent to the second Hilbert scheme of a K3 surface and the explicit construction of varieties realizing several of these maximal group actions.

The study of finite automorphism groups of K3 surfaces and more generally of hyperkähler manifolds has an over forty-year-long history resembling in several aspects the development around Monstrous Moonshine. A hyperkähler manifold is a $4n$-dimensional Riemannian manifold with the symplectic group $\mathrm{Sp}(n)$ (or smaller) as holonomy group. It has a family of complex structures parametrized by a two-sphere. This allows one to consider three different classes of finite isometry groups: Those which fix all complex structures, called *symplectic*, those which preserve a fixed complex structure, i.e. *holomorphic* ones, and those which do not necessarily fix any complex structure. One could argue that the first and the third class are the more natural ones to study. However, explicit examples are typically constructions from algebraic geometry. Thus, the second class, as studied in the thesis, may also be considered natural from this perspective. In any case, the symplectic automorphisms are the smallest and technically simplest class to study and this is probably the main reason why they were studied first.

For K3 surfaces, Mukai classified around 1988 the eleven maximal symplectic automorphism groups using the equivariant fixed-point formula and some finite group theory. The surprising result was that those are the maximal subgroups of the sporadic Mathieu group $M_{23}$ fixing exactly 5 points under its natural action on 24 points. He also provided explicit algebraic geometric constructions realizing each of the eleven groups. From the Torelli theorem and Nikulin's earlier work, it was in principle clear that for K3 surfaces many questions regarding finite automorphism groups can be reduced to questions of automorphisms of lattices, in particular the Leech lattice. However, this approach became only fully feasible with the advancement of computer algebra, culminating in my 2016 paper with Geoffrey Mason in which we determined all 230 fixed-point lattices of the Leech lattice. Hashimoto determined in 2012 the deformation equivalence classes of symplectic group actions and Kondo already established in 1999 the size of the largest holomorphic automorphism group of a K3 surface. Based on Hashimoto's results (now best proven using the mentioned result about the fixed-point lattices of the Leech lattice), Brandhorst and Hashimoto determined in 2023 the 4167 deformation equivalence classes of holomorphic automorphism groups of K3 surfaces.

Although only indirectly relevant for the thesis, the analogy between Monstrous Moonshine and automorphisms of hyperkähler manifolds is deeper. Instead of holomorphic automorphisms one may consider autoequivalences of the derived category of a K3 surface or, in more physical terms, the symmetries of the associated $N=(4,4)$ supersymmetric sigma models. This led to the yet not fully understood 2011 discovery of Mathieu Moonshine for the Mathieu group $M_{24}$. Monstrous Moonshine was explained by Borcherds using bosonic string theory in dimension $26 = 24 + 2$ whereas super string theory works for dimension $10 = 8+2$. It seems a big mystery why the numbers 8 and 24 appear in string theory, in the context of sphere packings (the sphere packing problem is only solved in dimensions 1, 2, 3, 8 and 24) and in the context of hyperkähler geometry: All *known* hyperkähler manifolds have an indefinite $L = H^2(X, \mathbf{Z})$ (Beauville-Bogomolov) lattice related either to the Leech lattice or to the much simpler to handle $E_8$ lattice. This fact — together with the Torelli theorem and the surjectivity of the period map for hyperkähler manifolds — allows one to study the finite automorphism groups of hyperkähler manifolds starting with our result about fixed-point lattices of the Leech lattice.

The author investigates the next interesting case of hyperkähler manifolds besides K3 surfaces, namely the case of fourfolds deformation equivalent to the second Hilbert scheme of a K3 surface. This case already resembles most of the properties one has or expects for the remaining known hyperkähler manifolds. The case of symplectic group actions was first studied by Mongardi in his 2013 Ph.D. thesis, establishing the main strategy for approaching the problem of finding all finite symplectic automorphisms groups. In my 2014 paper with Mason (finally published in 2019) we determined all deformation equivalence classes of maximal symplectic group actions: There are 26 cases corresponding to 15 different groups $G$. The approach was based in part on the equivariant fixed-point formula and finite group theory and in part on lattice calculations. It is now possible to use only the fixed-point lattices of the Leech lattice instead of the fixed-point formulas and finite group theory.

For holomorphic group actions one has to find conjugacy classes of groups $\widetilde{G}$ inside $O(L)$ which fit into a short exact sequence

$$1 \longrightarrow G \longrightarrow \widetilde{G} \longrightarrow \mu_m \longrightarrow 1$$

where $\mu_m$ is a cyclic group. The main technical insight is Lemma 3.2.4 which reduces the problem of finding conjugacy classes of groups $\widetilde{G}$ in the orthogonal group $O(L)$ to finding conjugacy classes of elements in the orthogonal group of the fixed-point lattice $L^G$ which are coming from symmetries of the whole lattice $L$. Starting from the 26 cases from my 2014 paper, the result of the computation is the Table 3.2 with 65 entries, one of the two main results of the thesis. I have verified the number of entries and most of the entries of that table with my computational tools I had developed since that time, starting from the Leech lattice. Thus I am very confident that the result which is based on computer calculations is correct.

The second main result of the thesis discusses explicit models of hyperkähler manifolds of $K3^{[2]}$ type realizing some of the found 65 cases. Some examples are generalizations of known construction methods to further groups (Section 4.2: Newly constructed EPW sextics). Section 4.3. considers Hilbert squares of quartics in $\mathbf{P}^3$. Although this construction is in principle known, the full automorphism group has to be identified. The next class are the so-called Beauville-Donagi fourfolds. The considered examples are known, but the full automorphism group has to be determined. The final example studied are Debarre-Voisin fourfolds which were also known before, but the full group of holomorphic automorphisms $2 \times L_2(11)$ and the correct entry in Table 3.2 has to be determined.

Overall, for about 14 of the 65 maximal entries in Table 3.2 concrete models were found. Sometimes it remains open to exactly which entry in the table one of the constructed examples belongs since there may be two possible options.

Some of the results of the thesis from Chapter 4 seem to be already published in joint work with coauthors. Thus I cannot exactly say which contributions are the original contributions of author of the thesis. I have not looked at the `Macaulay2` and `Magma` code in Appendices A.1, A.2 and A.3, but the calculations for Table 3.2 (Appendix A.2) I have mostly verified with my own programs.

The results obtained in the thesis are new, interesting, and contribute significantly to completing the picture of symmetries of known hyperkähler manifolds. The required computer calculations are non-trivial and not easy to implement. For me, Lemma 3.2.4 was the main missing theoretical insight to generate Table 3.2, though I am not certain if the lemma itself is new. While not entirely surprising, as they follow a path similar to many previous results for K3 surfaces, the results are nonetheless valuable. Computationally, one might also have considered determining the abstract isomorphism type of the groups $\widetilde{G}$. While this would offer a coarser classification than Table 3.2, identifying the isomorphism type could be more straightforward for the explicit examples. I noted a few obvious typographical errors.

I fully recommend acceptance of the doctoral thesis for the conferment of the doctoral degree without any revisions.

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