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Referee report for the PhD Thesis
Very symmetric hyper-Kähler fourfolds
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A complex manifold X of even dimension is called irreducible holomorphic or hyper-Kähler if X is Kähler, simply connected, compact, and $H^0(X,\Omega_X^2)$ is spanned by an everywhere non-degenerate two-form σ. $\mathrm{Aut}(X)$ will denote a group biholomorphic automorphisms of X (for short automorphisms). Moreover, by $\mathrm{Aut}_s(X)$ we shall denote a group of symplectic automorphisms. Recall that an automorphism of X is called symplectic if it preserves the symplectic form. The goal of this thesis is to study hyper-Kähler manifolds with a big finite group $\mathrm{Aut}(X)$. For any finite group action G on a hyper-Kähler manifold X, one can write the following exact sequence

$$1 \to \tilde{G} \to G \to \mu_m \to 1,$$

where μ_m is the group of m-th roots of unity for some natural m and $\tilde{G}$ is a subgroup of all the symplectic automorphisms in G. A hyper-Kähler manifold is of type $K3^{[2]}$ if its is deformation equivalent to the second Hilbert scheme of $K3$ surface. The goal of this thesis is the classification of finite groups of symplectic automorphisms acting on hyper-Kähler fourfolds of type $K3^{[2]}$. Among them, 15 are maximal and were classified in [HM19]. In the sequel, the author of the thesis classifies the polarized hyper-Kähler fourfolds of type $K3^{[2]}$ admitting the actions of finite groups G such that $\tilde{G}$ is one of the 15 maximal groups; we call such manifolds very symmetric. The main result of the thesis is the following classification.

Theorem (Th. 0.0.3) Let $\mathrm{Aut}_s(X) = SA$. There are 72 very symmetric hyper-Kähler fourfolds type $K3^{[2]}$ (X,G), in particular (1) $G \cap SA = L_2(11)$ in 6 cases, (2) $G \cap SA = L_3(4)$ in 6 cases, (3) $G \cap SA = \mathcal{A}_7$ in 10 cases, (4) $G \cap SA = \mathbb{Z}_2^3 : L_2(7)$ in 2 cases, (5) $G \cap SA = \mathbb{Z}_2 \times L_2(7)$ in 6 cases, (6) $G \cap SA = \mathbb{Z}_2^4 : \mathcal{A}_6$ in 6 cases, (7) $G \cap SA = \mathbb{Z}_2^4 : S_5$ in 6 cases, (8) $G \cap SA = S_6$ in 3 cases, (9) $G \cap SA = M_{10}$ in 8 cases, (10) $G \cap SA = (\mathbb{Z}_3 \times \mathcal{A}_5) : \mathbb{Z}_2$ in 6 cases, (11) $G \cap SA = \mathrm{Q}(\mathbb{Z}_3^2 : \mathbb{Z}_2)$ in 2 cases, (12) $G \cap SA = \mathbb{Z}_2^4 : (S_3 \times S_3)$ in

3 cases, (13) $G \cap SA = \mathbb{Z}_3^2 : \mathrm{Q}D_{16}$ in 2 cases, (14) $G \cap SA = 3^{1+4} : 2.2^2$ in 3 cases, (15) $G \cap SA = 3^4 : \mathcal{A}_6$ in 3 cases.

The proof of the above theorem is given in Chapter 3. The classification provided by Theorem 0.0.3 relies on the classification of finite groups of symplectic authomorphisms acting on hyper-Kähler fourfolds of type $K3^{[2]}$ obtained in [HM19]. To obtain their result Höhn and Mason used the following. Let L denote the $K3^{[2]}$ lattice.

Theorem 1 *Let G be a finite subgroup of $O(L)$ such that L_G is negative definite of rank at most 20, and it contains no square -2 vectors. Then there exists an embedding of $\mathcal{G}$-lattices*

$$L_G(-1), G) \hookrightarrow (\Lambda_{24}, Co_o)$$

such that $(\Lambda_{24})_G$ is the image.

Chapters 3 and 4 of [HM19] are devoted to establishing conditions for that to be possible. As originally shown in [Mon12], a finite action on the lattice L induced by a finite symplectic group action on some hyper-Kähler fourfold of type $K3^{[2]}$ satisfies the hypothesis of the above theorem. Because the Leech lattice is positive definite, computing its group actions becomes much simpler. The authors rely heavily on [MAGMA], to obtain their classification starting from there. The converse to the theorem does not hold, i.e. one cannot, in general, find for a subgroup $G \subset Co_0$, such that $((\Lambda_{24})_G, Co_0)$ is at most a 20-dimensional, appropriate action G on L. Sections 3 through 7 of [HM19] are devoted to establishing which subgroups can in fact be realized in this way. Having that, one finds all pairs (L_G, G) for G being a finite group action on L induced by a symplectic action on some hyper-Kähler fourfold of type $K3^{[2]}$. There are 15 maximal groups of this kind. In that case, $\mathrm{rk}L_G = 20$. Then $\mathrm{rk}L^G = 3$ and L^G is positive definite. There is only a finite number of possible isomorphism classes of rank three positive definite lattices that can be glued with any L_G to the lattice L. In [HM19] there is presented a list of 15 group. That means a list of 15 maximal groups acting symplectically on fourfolds of type $K3^{[2]}$ and the invariant sublattice. (See table 3.1.)

Consider a group G_s which is one of the 15 maximal finite groups which act on some hyper-Kähler fourfold of type $K3^{[2]}$ via symplectic automorphisms. The aim of the thesis under review is to analyze all finite groups G containing G_s as a subgroup for which there exists a hyper-Kähler fourfold of type $K3^{[2]}$ X, such that G has a representation as a subgroup of $Aut(X)$ and $G_s = G \cap Aut_s$ under this representation. Let us fix X, a projective hyper-Kähler fourfold of type $K3^{[2]}$ and $G_s \subset Aut_s(X)$ as a subgroup. Put $L = H^2(X, \mathbb{Z})$. From [HM19], we know what L^{G_s} and L_{G_s} might look like; L_{G_s} is fixed if we know G_s, but there might be multiple options for L^{G_s}. Nevertheless, they are all known. So let us fix one of them. In particular, we always have $\mathrm{rk}L^{G_s} = 3$, $\mathrm{rk}L_{G_s} = 20$ (cf. [HM19], Table 9), and L_{G_s} contains no wall divisors (cf. [HM19], Theorem 8.3). We also have the anti-embedding $\gamma : D_L^{G_s} \to D_L^{G_s}$ as in Lemma 3.2.1.

Let us assume moreover, that there exists $\hat{\mathrm{f}} \in Aut(X) \setminus G_s$ such that G, the group generated by $\hat{\mathrm{f}}$ and G_s is finite. By Lemma 3.2.2 and its proof, $\mathrm{f} = \hat{\mathrm{f}}_{|}L^{G_s}$ is *good*, and by knowing how it acts on L^{G_s}, we know the transcendental lattice as well as the primitive ample class fixed by G (there is necessarily exactly one such class). The last part of the proof of the main theorem presents a table 3.2 of groups with detailed explanations.

In the last chapters of the thesis the author provides a list of known explicit examples of very symmetric $K3^{[2]}$ type manifolds. Those examples are constructed as Fano varieties of lines on special cubic fourfolds as special Debarre-Voisin fourfolds or special EPW sextics. In particular, in Example 4.4.1 he describes the most symmetric $K3^{[2]}$ type manifold with a group of automorphisms of order 174960 as a Fano variety of lines on the Fermat cubic. Moreover, Section 4.3 proves the existence of birational models of Hilbert squares of two K3 surfaces that are very symmetric. To obtain his results, the author uses [GAP], [MAGMA] and [M2] systems for computations.

Special comments

1. page 8_2 "which lemma" ?
2. citation "Kuznetsov" without coordinates (Proposition 4.1.1) page 18
3. page 12 why Proposition 3.2.6 is without proof ?
4. where is the end of the proof of the main theorem 0.0.3 ?
5. some misprints:
   - what does "BB degree 2" mean on page 17^2 ?
   - what does "Table 3.2.2" mean on page 17^5 ?
   - first and second line of section 4.2 M_{10} and $\mathcal{A}_7$
   - third page of the Introduction: is 74960 should be 174960

Final Conclusions

I find the thesis very interesting. The mathematical results are very deep and difficult.

The dissertation fulfills the requirements for PhD theses in mathematical sciences formulated by law and accepted in the community.

I recommend mgr Tomasz Wawak to obtain the PhD degree.

Andrzej Szczepański