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# Very symmetric hyper-Kähler fourfolds-synopsis

Tomasz Wawak

Faculty of Mathematics and Computer Science,
Jagiellonian University

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**Abstract**

G. Höhn and G. Mason classified all finite groups acting faithfully and symplectically on a hyper-Kähler fourfold of type $K3^{[2]}$. There are 15 maximal among them, call them $G_s^1, \ldots, G_s^{15}$. We in turn classify the tuples $(X, h, G)$ where $(X, h)$ is a polarized hyper-Kähler fourfold of type $K3^{[2]}$, $G$ is a finite group acting faithfully on $X$ preserving $h$, $G$ contains $G_s^i$ as a proper subgroup for some $i$ so that $G_s^i$ acts symplectically on $X$. We also construct explicit examples of fourfolds of $K3^{[2]}$-type admitting an action of such groups.

The central problem of algebraic geometry is the classification of algebraic manifolds. We expect that a smooth projective manifold can be transformed into a minimal model, which means either a fibration in Fano varieties or a variety with a nef canonical class. This reduces the problem of classification to the problem of classifying minimal models. Varieties with trivial canonical class are minimal models and are situated between the well-studied Fano varieties and the vast set of varieties of general type.

**Definition.** A complex manifold X of even dimension is called irreducible holomorphic symplectic or hyper-Kähler if

1. $X$ is Kähler,
2. $X$ is simply connected, compact, and
3. $H^0(X, \Omega_X^2)$ is spanned by an everywhere non-degenerate two-form $\rho$.

Hyper-Kähler manifolds are always of even dimension; K3 surfaces are hyper-Kähler manifolds of dimension two. Note that in dimension 2 the first assumption follows from the second with the third. In higher dimensions the first assumption cannot be omitted. Hyper-Kähler manifolds are among the building blocks of Kähler manifolds with trivial canonical class.

**Theorem** (Beauville). *Let X be a compact Kähler manifold with trivial first Chern class. There exists a smooth finite cover $X'$ of $X$ isomorphic as a Kähler manifold to the finite product*

$$A \times \Pi CY_i \times \Pi HK_j$$

*where $A$ is a complex torus, $CY_i$ are compact Kähler varieties that are simply connected with holonomy group $SU(n_i)$ such that $\dim CY_i = n_i$, and $HK_j$ are compact Kähler varieties that are simply connected with holonomy group $Sp(m_j)$ and $\dim HK_j = 2m_j$.*

We call $CY_i$ Calabi–Yau varieties and we are interested in hyper-Kähler manifolds denoted $HK_j$ above. It is not known whether there is a finite number of families of hyper-Kähler manifolds in each dimension. In dimension 2, hyper-Kähler manifolds are called K3 surfaces. Kodaira proved that all K3 surfaces are deformation equivalent. In higher dimensions, much less is known about hyper-Kähler manifolds, and there are no general classification results. In each even dimension $2n$, we know of two families: the deformations of the Hilbert scheme $S^{[n]}$ of $n$ points on a K3 surface $S$, called type $K3^{[n]}$, and deformations of $K_n(T)$ the Hilbert scheme of $n+1$ points summing to 0 on an abelian surface $T$. Additionally, two sporadic families of such manifolds were constructed by O'Grady: OG6 type which is a deformation equivalent of desingularization of a moduli space of sheaves on a an Abelian surface and OG10 which is a deformation equivalent of desingularization of a moduli space of sheaves on a K3 surface.

The goal of the thesis is to study hyper-Kähler manifolds with a huge finite group of automorphisms. An automorphism of $X$ is called *symplectic* if it preserves the symplectic form, otherwise it is called *nonsymplectic*. Recall that the second cohomology group $\mathrm{H}^2(X,\mathbb{Z})$ has a natural lattice structure owing to the Beauville-Bogomolov form[(1)] (see [Bea83]). If $X$ is a K3 surface, this form induces the standard intersection pairing on the Picard group.

For any finite group action $G$ on a hyper-Kähler manifold X, one can write the following exact sequence

$$1 \to \widetilde{G} \to G \to \mu_m \to 1, \tag{1}$$

where $\mu_m$ is the group of $m$-th roots of unity for some natural $m$ and $\widetilde{G}$ is a subgroup of all the symplectic automorphisms in $G$. In the case of K3 surfaces, this simple observation along with some bounds for possible $m$ as seen in [Nik79] was one of the basic tools allowing the study of large finite groups of automorphisms of K3 surfaces after the classification of the symplectic ones by Mukai in [Muk88]. In particular Kondo in [Kon99] shows that the maximum order of a finite group of automorphisms of a K3 surface is 3840 and describe such a K3 surface as a Kummer surface. In the works [BS21, BH21] the authors describe other examples of interesting symmetric K3 surfaces.

The goal of the thesis is to follow this schema using the classification (obtained in [HM19]) of finite groups of symplectic automorphisms acting on hyper-Kähler fourfolds of type $\mathrm{K3}^{[2]}$. Among them, 15 are maximal. In the sequel, we classify the polarized hyper-Kähler fourfolds of type $\mathrm{K3}^{[2]}$ admitting the actions of finite groups $G$ such that $\widetilde{G}$ is one of the 15 maximal groups; we call such manifolds *very symmetric*. The classification is up to the transcendental lattice, the polarization type of the vector invariant under the action and the invariant lattice for the symplectic part of the group action; the results are presented in the table (cf. [BH21] in the case of K3 surfaces). The thesis is primarily based on the following artivles [Waw22, BMW24, BW24].

To attain our results, we use [GAP21],[MAGMA], and [M2] systems for computations.

## Overview of results

The main result of the thesis is the following classification of polarized hyper-Kähler fourfolds of type $\mathrm{K3}^{[2]}$.

| # | $(h^2, \mathrm{div}h)$ | $G_s$ | $m$ | $\mathrm{T}_X$ | $L^{G_s}$ | #var | K3 | ex |
|---|---|---|---|---|---|---|---|---|
| 1. | (2, 1) | $L_2(11)$ | 2 | $\begin{pmatrix}22&0\\0&22\end{pmatrix}$ | $\begin{pmatrix}2&1&0\\1&6&0\\0&0&22\end{pmatrix}$ | 1 | f | Y |
| 2. | (2, 1) | $L_3(4)$ | 2 | $\begin{pmatrix}10&4\\4&10\end{pmatrix}$ | $\begin{pmatrix}2&0&0\\0&10&4\\0&4&10\end{pmatrix}$ | 1 | - | - |
| 3. | (2, 1) | $A_7$ | 2 | $\begin{pmatrix}6&0\\0&70\end{pmatrix}$ | $\begin{pmatrix}2&1&0\\1&2&0\\0&0&70\end{pmatrix}$ | 1 | f | Y |
| 4. | (2, 1) | $A_7$ | 2 | $\begin{pmatrix}6&0\\0&70\end{pmatrix}$ | $\begin{pmatrix}2&0&1\\0&6&0\\1&0&18\end{pmatrix}$ | 1 | f | Y |
| 5. | (2, 1) | $\mathbb{Z}_2 \times L_2(7)$ | 4 | $\begin{pmatrix}14&0\\0&14\end{pmatrix}$ | $\begin{pmatrix}2&0&0\\0&14&0\\0&0&14\end{pmatrix}$ | 1 | t | Y |
| 6. | (2, 1) | $\mathbb{Z}_2^4 : A_6$ | 2 | $\begin{pmatrix}4&0\\0&24\end{pmatrix}$ | $\begin{pmatrix}2&0&0\\0&4&0\\0&0&24\end{pmatrix}$ | 1 | - | - |
| 7. | (2, 1) | $\mathbb{Z}_2^4 : S_5$ | 2 | $\begin{pmatrix}4&0\\0&40\end{pmatrix}$ | $\begin{pmatrix}2&0&0\\0&4&0\\0&0&40\end{pmatrix}$ | 1 | t | Y |

[(1)]Sometimes also called Fujiki-Beauville-Bogomolov.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8. | (2, 1) | $M_{10}$ | 2 | $\begin{pmatrix}4 & 0\\0 & 30\end{pmatrix}$ | $\begin{pmatrix}2 & 0 & 0\\0 & 4 & 0\\0 & 0 & 30\end{pmatrix}$ | 1 | f | Y/-[(2)] |
| 9. | (2, 1) | $M_{10}$ | 2 | $\begin{pmatrix}4 & 0\\0 & 30\end{pmatrix}$ | $\begin{pmatrix}2 & 0 & 0\\0 & 4 & 0\\0 & 0 & 30\end{pmatrix}$ | 1 | - | Y/-[(3)] |
| 10. | (4, 1) | $L_3(4)$ | 2 | $\begin{pmatrix}12 & 0\\0 & 14\end{pmatrix}$ | $\begin{pmatrix}4 & 2 & 0\\2 & 4 & 0\\0 & 0 & 14\end{pmatrix}$ | 1 | f | - |
| 11. | (4, 1) | $\mathbb{Z}_2^3 : L_2(7)$ | 2 | $\begin{pmatrix}6 & 2\\2 & 10\end{pmatrix}$ | $\begin{pmatrix}4 & 0 & 0\\0 & 6 & 2\\0 & 2 & 10\end{pmatrix}$ | 2 | - | - |
| 12. | (4, 1) | $\mathbb{Z}_2 \times L_2(7)$ | 2 | $\begin{pmatrix}14 & 0\\0 & 28\end{pmatrix}$ | $\begin{pmatrix}4 & 2 & 0\\2 & 8 & 0\\0 & 0 & 14\end{pmatrix}$ | 1 | f | - |
| 13. | (4, 1) | $\mathbb{Z}_2^1 : A_6$ | 2 | $\begin{pmatrix}2 & 0\\0 & 24\end{pmatrix}$ | $\begin{pmatrix}2 & 0 & 0\\0 & 4 & 0\\0 & 0 & 24\end{pmatrix}$ | 1 | - | - |
| 14. | (4, 1) | $\mathbb{Z}_2^1 : A_6$ | 2 | $\begin{pmatrix}6 & 0\\0 & 8\end{pmatrix}$ | $\begin{pmatrix}4 & 0 & 0\\0 & 6 & 0\\0 & 0 & 8\end{pmatrix}$ | 1 | f | - |
| 15. | (4, 1) | $\mathbb{Z}_2^1 : S_5$ | 2 | $\begin{pmatrix}2 & 0\\0 & 40\end{pmatrix}$ | $\begin{pmatrix}2 & 0 & 0\\0 & 4 & 0\\0 & 0 & 40\end{pmatrix}$ | 1 | - | - |
| 16. | (4, 1) | $\mathbb{Z}_2^1 : S_5$ | 2 | $\begin{pmatrix}8 & 0\\0 & 10\end{pmatrix}$ | $\begin{pmatrix}4 & 0 & 0\\0 & 8 & 0\\0 & 0 & 10\end{pmatrix}$ | 1 | - | - |
| 17. | (4, 1) | $S_6$ | 2 | $\begin{pmatrix}12 & 0\\0 & 30\end{pmatrix}$ | $\begin{pmatrix}4 & 2 & 0\\2 & 4 & 0\\0 & 0 & 30\end{pmatrix}$ | 1 | f | - |
| 18. | (4, 1) | $M_{10}$ | 2 | $\begin{pmatrix}2 & 0\\0 & 30\end{pmatrix}$ | $\begin{pmatrix}2 & 0 & 0\\0 & 4 & 0\\0 & 0 & 30\end{pmatrix}$ | 1 | f | - |
| 19. | (4, 1) | $M_{10}$ | 2 | $\begin{pmatrix}2 & 0\\0 & 30\end{pmatrix}$ | $\begin{pmatrix}2 & 0 & 0\\0 & 4 & 0\\0 & 0 & 30\end{pmatrix}$ | 1 | - | - |
| 20. | (4, 1) | $\mathbb{Z}_2^1 : (S_3 \times S_3)$ | 2 | $\begin{pmatrix}6 & 0\\0 & 24\end{pmatrix}$ | $\begin{pmatrix}4 & 0 & 0\\0 & 6 & 0\\0 & 0 & 24\end{pmatrix}$ | 1 | f | - |
| 21. | (6, 1) | $A_7$ | 2 | $\begin{pmatrix}2 & 0\\0 & 70\end{pmatrix}$ | $\begin{pmatrix}2 & 1 & 0\\1 & 2 & 0\\0 & 0 & 70\end{pmatrix}$ | 1 | f | - |
| 22. | (6, 1) | $A_7$ | 2 | $\begin{pmatrix}8 & 2\\2 & 18\end{pmatrix}$ | $\begin{pmatrix}6 & 3 & 1\\3 & 6 & 1\\1 & 1 & 8\end{pmatrix}$ | 2 | f | - |
| 23. | (6, 1) | $\mathbb{Z}_2^1 : A_6$ | 2 | $\begin{pmatrix}4 & 0\\0 & 8\end{pmatrix}$ | $\begin{pmatrix}4 & 0 & 0\\0 & 6 & 0\\0 & 0 & 8\end{pmatrix}$ | 1 | - | - |
| 24. | (6, 1) | $(\mathbb{Z}_3 \times A_5) : \mathbb{Z}_2$ | 2 | $\begin{pmatrix}10 & 0\\0 & 30\end{pmatrix}$ | $\begin{pmatrix}4 & 1 & 0\\1 & 4 & 0\\0 & 0 & 30\end{pmatrix}$ | 1 | f | - |
| 25. | (6, 1) | $\mathbb{Z}_2^1 : (S_3 \times S_3)$ | 2 | $\begin{pmatrix}4 & 0\\0 & 24\end{pmatrix}$ | $\begin{pmatrix}4 & 0 & 0\\0 & 6 & 0\\0 & 0 & 24\end{pmatrix}$ | 1 | - | - |
| 26. | (6, 1) | $3^{1+1} : 2.2^2$ | 2 | $\begin{pmatrix}6 & 0\\0 & 6\end{pmatrix}$ | $\begin{pmatrix}6 & 0 & 0\\0 & 6 & 0\\0 & 0 & 6\end{pmatrix}$ | 1 | f | - |
| 27. | (6, 1) | $3^1 : A_6$ | 2 | $\begin{pmatrix}6 & 0\\0 & 18\end{pmatrix}$ | $\begin{pmatrix}6 & 3 & 0\\3 & 6 & 0\\0 & 0 & 6\end{pmatrix}$ | 1 | f | - |

[(2)] At least one out of this one and the following one is known, but we cannot determine whether it is both, just one, and in that case, which.

[(3)] At least one out of this one and the following one is known, but we cannot determine whether it is both, just one, and in that case, which.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 28. | (6, 2) | $L_2(11)$ | 3 | $\begin{pmatrix} 22 & 11 \\ 11 & 22 \end{pmatrix}$ | $\begin{pmatrix} 6 & 2 & 2 \\ 2 & 8 & -3 \\ 2 & -3 & 8 \end{pmatrix}$ | 1 | - | Y |
| 29. | (6, 2) | $A_7$ | 2 | $\begin{pmatrix} 2 & 1 \\ 1 & 18 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 6 & 0 \\ 1 & 0 & 18 \end{pmatrix}$ | 1 | - | Y |
| 30. | (6, 2) | $(\mathbb{Z}_3 \times A_5) : \mathbb{Z}_2$ | 6 | $\begin{pmatrix} 10 & 5 \\ 5 & 10 \end{pmatrix}$ | $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 10 & 5 \\ 0 & 5 & 10 \end{pmatrix}$ | 1 | - | Y |
| 31. | (6, 2) | $3^{1+4} : 2.2^2$ | 4 | $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$ | $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ | 1 | - | Y |
| 32. | (6, 2) | $3^4 : A_6$ | 6 | $\begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}$ | $\begin{pmatrix} 6 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ | 1 | - | Y |
| 33. | (8, 1) | $\mathbb{Z}_2^4 : A_6$ | 2 | $\begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$ | $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ | 1 | - | - |
| 34. | (8, 1) | $\mathbb{Z}_2^4 : S_5$ | 2 | $\begin{pmatrix} 4 & 0 \\ 0 & 10 \end{pmatrix}$ | $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 10 \end{pmatrix}$ | 1 | f | - |
| 35. | (10, 1) | $\mathbb{Z}_2^4 : S_5$ | 2 | $\begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}$ | $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 10 \end{pmatrix}$ | 1 | - | - |
| 36. | (10, 1) | $(\mathbb{Z}_3 \times A_5) : \mathbb{Z}_2$ | 2 | $\begin{pmatrix} 6 & 0 \\ 0 & 30 \end{pmatrix}$ | $\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 30 \end{pmatrix}$ | 1 | f | - |
| 37. | (10, 1) | $(\mathbb{Z}_3 \times A_5) : \mathbb{Z}_2$ | 2 | $\begin{pmatrix} 6 & 0 \\ 0 & 30 \end{pmatrix}$ | $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 10 & 5 \\ 0 & 5 & 10 \end{pmatrix}$ | 1 | f | - |
| 38. | (12, 1) | $L_3(4)$ | 2 | $\begin{pmatrix} 2 & 0 \\ 0 & 28 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 10 & 4 \\ 0 & 4 & 10 \end{pmatrix}$ | 1 | f | - |
| 39. | (12, 1) | $L_3(4)$ | 2 | $\begin{pmatrix} 4 & 0 \\ 0 & 14 \end{pmatrix}$ | $\begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 14 \end{pmatrix}$ | 1 | f | - |
| 40. | (12, 1) | $S_6$ | 2 | $\begin{pmatrix} 4 & 0 \\ 0 & 30 \end{pmatrix}$ | $\begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 30 \end{pmatrix}$ | 1 | f | - |
| 41. | (12, 1) | $M_{10}$ | 2 | $\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}$ | $\begin{pmatrix} 4 & 2 & 0 \\ 2 & 6 & 0 \\ 0 & 0 & 12 \end{pmatrix}$ | 2 | f | - |
| 42. | (12, 1) | $\mathbb{Z}_3^2 : QD_{16}$ | 2 | $\begin{pmatrix} 4 & 2 \\ 2 & 10 \end{pmatrix}$ | $\begin{pmatrix} 4 & 2 & 0 \\ 2 & 10 & 0 \\ 0 & 0 & 12 \end{pmatrix}$ | 2 | - | - |
| 43. | (12, 1) | $3^{1+4} : 2.2^2$ | 2 | $\begin{pmatrix} 6 & 0 \\ 0 & 12 \end{pmatrix}$ | $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ | 1 | f | - |
| 44. | (14, 1) | $\mathbb{Z}_2 \times L_2(7)$ | 2 | $\begin{pmatrix} 2 & 0 \\ 0 & 14 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{pmatrix}$ | 1 | - | - |
| 45. | (14, 2) | $L_3(4)$ | 6 | $\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$ | $\begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 14 \end{pmatrix}$ | 1 | - | - |
| 46. | (14, 2) | $\mathbb{Z}_2 \times L_2(7)$ | 2 | $\begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix}$ | $\begin{pmatrix} 4 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 14 \end{pmatrix}$ | 1 | - | - |
| 47. | (16, 1) | $Q(\mathbb{Z}_3^2 : \mathbb{Z}_2)$ | 2 | $\begin{pmatrix} 8 & 4 \\ 4 & 14 \end{pmatrix}$ | $\begin{pmatrix} 6 & 2 & 2 \\ 2 & 6 & -2 \\ 2 & -2 & 14 \end{pmatrix}$ | 2 | f | - |
| 48. | (18, 1) | $3^4 : A_6$ | 2 | $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$ | $\begin{pmatrix} 6 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ | 1 | f | - |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 49. | (22, 1) | $L_2(11)$ | 2 | $\begin{pmatrix} 2 & 0 \\ 0 & 22 \end{pmatrix}$ | $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 22 \end{pmatrix}$ | 1 | f | - |
| 50. | (22, 1) | $L_2(11)$ | 2 | $\begin{pmatrix} 6 & 2 \\ 2 & 8 \end{pmatrix}$ | $\begin{pmatrix} 6 & 2 & 2 \\ 2 & 8 & -3 \\ 2 & -3 & 8 \end{pmatrix}$ | 2 | f | - |
| 51. | (22, 2) | $L_2(11)$ | 2 | $\begin{pmatrix} 2 & 1 \\ 1 & 6 \end{pmatrix}$ | $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 22 \end{pmatrix}$ | 1 | t | Y |
| 52. | (24, 1) | $\mathbb{Z}_2^4 : A_6$ | 2 | $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 24 \end{pmatrix}$ | 1 | - | - |
| 53. | (24, 1) | $\mathbb{Z}_2^4 : (S_3 \times S_3)$ | 2 | $\begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$ | $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 24 \end{pmatrix}$ | 1 | - | - |
| 54. | (28, 1) | $L_3(4)$ | 2 | $\begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 10 & 4 \\ 0 & 4 & 10 \end{pmatrix}$ | 1 | f | - |
| 55. | (28, 1) | $\mathbb{Z}_2 \times L_2(7)$ | 2 | $\begin{pmatrix} 2 & 0 \\ 0 & 28 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{pmatrix}$ | 1 | f | - |
| 56. | (28, 1) | $\mathbb{Z}_2 \times L_2(7)$ | 2 | $\begin{pmatrix} 4 & 0 \\ 0 & 14 \end{pmatrix}$ | $\begin{pmatrix} 4 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 14 \end{pmatrix}$ | 1 | f | - |
| 57. | (30, 1) | $M_{10}$ | 2 | $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 30 \end{pmatrix}$ | 1 | f | - |
| 58. | (30, 1) | $(\mathbb{Z}_3 \times A_5) : \mathbb{Z}_2$ | 2 | $\begin{pmatrix} 6 & 0 \\ 0 & 10 \end{pmatrix}$ | $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 10 & 5 \\ 0 & 5 & 10 \end{pmatrix}$ | 1 | f | - |
| 59. | (30, 2) | $S_6$ | 2 | $\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$ | $\begin{pmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 30 \end{pmatrix}$ | 1 | - | - |
| 60. | (30, 2) | $M_{10}$ | 2 | $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 30 \end{pmatrix}$ | 1 | - | - |
| 61. | (30, 2) | $(\mathbb{Z}_3 \times A_5) : \mathbb{Z}_2$ | 2 | $\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ | $\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 30 \end{pmatrix}$ | 1 | - | - |
| 62. | (40, 1) | $\mathbb{Z}_2^4 : S_5$ | 2 | $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 40 \end{pmatrix}$ | 1 | - | - |
| 63. | (42, 1) | $A_7$ | 2 | $\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}$ | $\begin{pmatrix} 4 & 2 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 12 \end{pmatrix}$ | 2 | f | - |
| 64. | (70, 1) | $A_7$ | 2 | $\begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$ | $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 6 & 0 \\ 1 & 0 & 18 \end{pmatrix}$ | 1 | f | - |
| 65. | (70, 2) | $A_7$ | 6 | $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ | $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 70 \end{pmatrix}$ | 1 | t | - |

Table 1: Representations of the group actions

The first column describes the polarization $h$ in the Picard group, where $h^2$ is the value of the Beauville-Bogomolov form on $h$ and div$h$ is its divisibility as a lattice element. The second is the $G_s$ as in 1, $m$ is as in the $\mu_m$ in 1 and together they describe $G$ the largest finite group action on $X$ respecting the polarization $h$. The $\mathrm{T}_X$ and $L^{G_s}$ are the Gram matrices of respectively the transcendental lattice of $X$ i.e. the orthogonal component of the Picard group of $X$ in $\mathrm{H}^2(X, \mathbb{Z})$ and the invariant sublattice of $\mathrm{H}^2(X, \mathbb{Z})$ under the action of $G_s$. The column #var tells how many polarized varieties are described in the given row (they are birational). The column K3 contains an information on whether a variety is birational to Hilbert scheme of points on some K3 rather than just being a deformation ("t" is confirmed birational, "f" confirmed non-birational, "-" unknown). Finally, the last column explains whether an explicit construction is known.

## Explicit constructions

EPW-sextics are singular sextic hypersurfaces in $\mathbb{P}^5$, firstly constructed by Eisenbud, Popescu and Walter ([EPW01]). They come provided with a natural double cover, and O'Grady showed that the cover of a generic EPW is a hyper-Kähler that is deformation equivalent to the Hilbert square of a K3 surface ([O'G06]). We call this double cover a double EPW sextic. We briefly recall the construction.

Let us fix once and for all a 6-dimensional $\mathbb{C}$-vector space $V_6$ with a volume form $vol\colon \bigwedge^6 V_6 \xrightarrow{\cong} \mathbb{C}$. Consider the sub vector-bundle $F \subset \bigwedge^3 V_6 \otimes \mathcal{O}_{\mathbb{P}(V_6)}$ whose fiber over $[v] \in \mathbb{P}(V_6)$ is given by:

$$F_v = \{\alpha \in \bigwedge^3 V_6;\ \alpha \wedge v = 0\}.$$

Since the symplectic form is zero on $F$ and $2\dim(F_v) = 20 = \dim(\bigwedge^3 V_6)$, the sub vector-bundle is Lagrangian for the symplectic form induced by the one on $\bigwedge^3 V_6$. Let $\mathbb{LG}(\bigwedge^3 V_6)$ be the Lagrangian Grassmannian parameterizing the Lagrangian subspaces, and fix an element $A \in \mathbb{LG}(\bigwedge^3 V_6)$.
Define degeneracy loci

$$Y_A[k] = \{[v] \in \mathbb{P}(V_6) | \dim(A \cap F_v) \geq k\}.$$

We call the locus $Y_A = Y_A[1]$ an EPW sexitc. We say that $A$ does not contain decomposable vectors if it has no nonzero vectors of the form $x \wedge y \wedge z$ (i.e. $\mathbb{P}(A) \cap \mathrm{Gr}(3, V_6) = \emptyset$ in $\mathbb{P}(\bigwedge^3 V_6)$). In this section we will always assume that $A$ has no decomposable vectors. Then $Y_A$ is always a sextic hypersurface. An important fact about automorphisms of $Y_A$ is that

$$\mathrm{Aut}(Y_A) = \{g \in \mathrm{PGL}(V_6) | (\bigwedge^3 g)(A) = A\} \tag{2}$$

and this is a finite group by ([DK18, Proposition B.9]).
Let us denote

$$\mathbb{LG}(\bigwedge^3 V_6)^0 = \{A \in \mathbb{LG}(\bigwedge^3 V_6) \mid \mathbb{P}(A) \cap \mathrm{Gr}(3, V_6) = \emptyset,\ Y_A[3] = \emptyset\} =$$
$$= \{A \in \mathbb{LG}(\bigwedge^3 V_6) \mid \mathrm{Sing}\, Y_A = Y_A[2],\ \mathrm{Sing}\, Y_A[2] = Y_A[3] = \emptyset\}.$$

For $A$ from the above subset (which is open in the Lagrangian Grassmanian), there is a canonical double cover $\pi_A : \widetilde{Y}_A \to Y_A$ branched along the surface $Y_A[2]$, which is of great interest for us since for a generic $A$ it is a hyper-Kähler deformation equivalent to the Hilbert square of a K3 surface ([O'G06]). $\widetilde{Y}_A$ carries a canonical polarization $H = \pi_A^* \mathcal{O}_{Y_A}(1)$ and the image of the morphism $\widetilde{Y}_A \to \mathbb{P}(\mathrm{H}^0(\widetilde{Y}_A, H)^\vee)$ is isomorphic to $Y_A$.
Every automorphism of $Y_A$ induces an automorphism of $\widetilde{Y}_A$ that fixes the class $H$ (proof of [DK18, Proposition B.8(b)]), conversely any automorphism of $\widetilde{Y}_A$ that fixes $H$ induces an isomorphism $\mathbb{P}(\mathrm{H}^0(\widetilde{Y}_A, H)^\vee) \cong \mathbb{P}(V_6)$ hence descends to an automorphism of $Y_A$. Denote by $\mathrm{Aut}_H(\widetilde{Y}_A)$ the group of automorphisms that fix the class $H$ and by $\iota$ the covering involution of $\pi_A$. The discussion above gives a central extension

$$1 \to \langle \iota \rangle \to \mathrm{Aut}_H(\widetilde{Y}_A) \to \mathrm{Aut}(Y_A) \to 1, \tag{3}$$

moreover denoting by $\mathrm{Aut}_H^s(\widetilde{Y}_A)$ the subgroup of $\mathrm{Aut}_H(\widetilde{Y}_A)$ consisting of symplectic automorphisms, one gets an extension

$$1 \to \mathrm{Aut}_H^s(\widetilde{Y}_A) \to \mathrm{Aut}_H(\widetilde{Y}_A) \to \mu_r \to 1 \tag{4}$$

with $\mu_r$ a finite group of order $r$. Note that the image of $\iota$ in $\mu_r$ is given by $-1$.

**Proposition 0.1** (Kuznetsov). *Let $A \subset \bigwedge^3 V_6$ be a Lagrangian subspace with no decomposable vectors. Then the extensions (3) and (4) are trivial and $r = 2$. In particular there is an isomorphism*

$$\mathrm{Aut}_H(\widetilde{Y}_A) \cong \mathrm{Aut}(Y_A) \times \langle \iota \rangle$$

*which splits (3) and the factor $\mathrm{Aut}(Y_A)$ corresponds to the subgroup $\mathrm{Aut}^s_H(\widetilde{Y}_A)$.*

In [BW24], a joint work with Simone Billi, we constructed two double EPW sextics with a symplectic action of $A_7$. Separately, I constructed such examples for manifolds with actions of $L_3(4)$ and $M_{10}$ (these were later the basis for the construction of new hyper-Kähler manifolds in [BMW24]).

The general idea is to start from a representation of a given group $\mathbb{P}(V_6)$, consider the third wedge of the representation, check if it is decomposable, and the summands are Lagangian, and lastly verify these summands belong to $\mathbb{LG}(\bigwedge^3 V_6)^0$.

Separately, we also construct other original examples as birational models of Hilbert squares of quartics in $\mathbb{P}^3$, the very symmetric hyper-Kähler fourfolds of type $K3^{[2]}$ with symplectic actions of groups $\mathbb{Z}_2 \times L_2(7)$ and $\mathbb{Z}_2^4 : S_5$

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