Report on the doctoral thesis "Selected interpolation nodes for polynomial approximation" by Dimitri Jordan Kenne

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The main topic of the present thesis concerns the classical problem of finding interpolations nodes which yield good polynomial approximation properties in multivariate setting. This problem has been very thoroughly investigated and documented in univariate case, on the other hand the considerably harder and more complex case of several variables presents an important research area which stayed in the center of attention up to the present day.

The thesis consists of 7 chapters and an extensive bibliography. The first introductory chapter gives a comprehensive overview of the main notions and tools needed in the sequel. In particular this includes definitions and main properties of transfinite diameter, pluricomplex Green function, equilibrium measure, Fekete and Leja points, admissible meshes. This well written introduction shows a deep understanding of the concepts and machinery required for a successful research work.

In Chapters 2 and 3 multivariate pseudo Leja sequences are introduces and studied, while Chapter 4 concerns computation of pseudo Leja points. In Chapter 5 the author uses the concept of admissible meshes for extracting Fekete-like and Leja-like interpolation sets, and evaluating their Lebesgue constants on piecewise polynomial or trigonometric curves in the complex plane. In Chapter 6 certain products of univariate admissible polynomial meshes are used in order to evaluate Lebesgue constants on cubes, simplices, and balls. Finally, Chapter 7 contains an interesting new lower bound for the transfinite diameter of Bernstein sets, which is derived based on application of Leja points. The main new results of the thesis are publishes in papers [1] and [2] written solely by the author and articles [3],[4] written with collaborators.

Chapter 2 is devoted to the introduction and study of multivariate pseudo Leja sequences, it contains main bulk of the new results. Here the author first introduces the concept of multivariate pseudo Leja sequences by inserting a proper Erdei type growth condition into the size of the Vandermonde determinants used for the definition. This definition shows some similarity to the notion of Fekete points, so it naturally leads to establishing the corresponding intrinsic relation between pseudo Leja points and multivariate transfinite diameter and equilibrium measure of the domain, see Theorem 2.2.1 and Corollary 2.2.2. The significance of the material given in Chapter 2 consists in the fact that *it opens the door to the future research.* The main unanswered question here which deserves future investigation concerns the size of the Lebesgue constants for multivariate pseudo Leja sequences. In a remarkable recent paper by V.Totik "The Lebesgue constants for Leja points are subexponential", J. Approx. Theory, 2023 it was shown that univariate pseudo Leja sequences of constant Erdei growth have Lebesgue constants of subexponential growth. This naturally yields the convergence of corresponding interpolation processes for holomorphic functions. In view of the results of Chapter 2 it would be interesting to verify subexponential growth of Lebesgue constants for multivariate pseudo Leja sequences. Can it be done at least for some model multivariate domains?

Chapter 3 is concerned with construction of multivariate pseudo Leja sequences for product spaces. Clearly, here one has to come up with some delicate intertwining technique in order to construct a pseudo Leja sequence based on two given sequences. Generalizing an approach proposed earlier by J.- P. Calvi the author accomplishes this for $\mathbf{C}^n \times \mathbf{C}^m$ product spaces, see Theorem 3.2.1. Moreover, similar intertwining technique is applied later on in order to verify convergence of interpolation processes based on intertwining pseudo Leja sequences extracted from disconnected compact planar sets (Theorem 3.3.3).

Chapter 4 is devoted to the computation of multivariate pseudo Leja points based on application of weakly admissible meshes. It includes examples of their numerical computation.

Admissible polynomial meshes are know to provide a good basis for constructing Fekete and Leja type sequences used in polynomial interpolation. This fact is the motivation for the material presented in Chapters 5 and 6, which discuss construction of admissible meshes having asymptotically optimal cardinality for certain classes of complex curves and domains, and similarly on the cube, simplex and the ball in $\mathbb{R}^n$. Then the author gives estimates for the Lebesgue constants related to these meshes. The theoretical aspects of these two chapters are somewhat standard, the presented material has mostly numerical value.

In Chapter 7 the author improves certain earlier lower estimates given for the transfinite diameter of Bernstein sets. It is shown in Theorem 7.1.1. that for every compact set K in $\mathbf{C}^n$ which has the Bernstein property (i.e., satisfies the Bernstein polynomial inequality) with parameter M its transfinite diameter $D(K)$ satisfies the lower bound

$$D(K) \geq \frac{1}{nM}.$$

The above estimate is quite elegant since it turns into equality in the univariate case $n = 1$. On the other hand it seems that the "curse of dimension" comes into play in the above inequality since its right hand side tends to 0 as $n \to \infty$. An intriguing question arises in this connection: is it possible to give examples of multivariate Bernstein domains for which $D(K)M$ approaches 0 as $n \to \infty$?

Summarizing, it appears to the reviewer that the author of the present thesis exhibited a deep understanding of the theory and methods of multivariate interpolation theory, and showed his ability to use this knowledge in order to obtain new significant results. Therefore based on this work the author certainly deserves to be awarded the doctoral degree.

References

[1] D. J. Kenne, *Multidimensional pseudo Leja sequences,* Dol. Res. Notes Approx., **17**, 2024.

[2] D. J. Kenne, *A New estimate of the transfinite diameter of Bernstein sets,* Math. Ineq. Appl., **27**, 2024.

[3] L. Białas-Cież, D. J. Kenne, A. Sommariva, and M. Vianello, *Chebyshev admissible meshes and Lebesgue constants of complex polynomial projections,* J. Comput. Appl. Math., **443**, 2024.

[4] L. Bialas-Ciez, D. J. Kenne, A. Sommariva, and M. Vianello, *Evaluating Lebesgue constants by Chebyshev polynomial meshes on cube, simplex and ball,* Electron. Trans. Numer. Anal, **60**, 2024.

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January 27, 2025, Budapest