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REPORT ON PHD THESIS OF DIMITRI JORDAN KENNE

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Dimitri Jordan Kenne submitted thesis entitled *Selected interpolation nodes for polynomial approximation*, written under supervision of dr hab. Leokadia Białas-Cież, professor of Jagiellonian University.

The thesis is in the area of complex analysis of one and several variables and it focuses mainly on specific questions of polynomial approximation, with examples of numerical implementation of proposed methods.

Exploiting the tools developed in the thesis, the last chapter, of somewhat different character, provides a new, improved estimate for the transfinite diameter of Bernstein sets.

The questions considered in the paper have a long tradition. The work refers to classical seminal works of Franciszek Leja and Józef Siciak, followed by subsequent, strong and fruitful works of members of Complex Analysis group in Jagiellonian University. In particular, the advisor of the thesis, professor L.Białas- Cież is an active member of the group, and a part of the thesis (namely: chapters 5 and 6) is based on joint work with the advisor.

The candidate is an author of four published research articles; two of them were written jointly with L. Białas- Cież, A. Sommariva, and M. Vianello.

Moreover, the candidate worked jointly with A. Sommariva nad M. Vianello on Matlab and Python codes used for various question of complex approximation; these codes have been also published and are freely available.

D. Kenne took part in four international conferences, and presented his results at these conferences.

The dissertation is well structured into seven chapters.

Chapter 1, Preliminaries, introduces the reader into the main subject and methods. This introductory chapter is very well written. I would also like to emphasize that the presentation is equipped with a rich and carefully prepared collection of references regarding particular items discussed in the paper.

Chapter 2 is devoted to the notion of pseudo- Leja sequences, in higher dimension. The question about such sequences is very natural, since, on one hand, it is a natural extension of Leja sequences, and, on the other hand, in the recent paper [11] pseudo Leja sequences in dimension 1 have been studied, and their properties similar to those of Leja sequneces have been descirbed. It is also natural to try to extend the (quite rigid) notion of Leja sequences to (more flexible) notion of pseudo- Leja sequences, because of crucial role of these objects in approximation problems (also, for numerical approach to these questions).

The main result (a analogue of the corresponding fact in previously studied settings) is Theorem 2.2.1, which shows that the transfinite diameter of a polynomially determining compact set can be approximated in terms of pseudo- Leja sequence. Also, the equilibrium measure can be obtained as a limit of natural sequence of equally distributed point masses on (suitably chosen) initial segments of pseudo- Leja sequence.

As a corollary, one obtains that also the pluricomplex Green function of a polynomially convex regular compact subset of $\mathbb{C}^n$ can be obtained as an (upper) limit of natural functions expressed in terms of pseudo-Leja sequence.

Even though the proofs are (as the author admits) rather natural extensions of classical ones, the results of this chapter are, in my opinion, relevant.

The content of Chapter 2 is a part of the paper written by the candidate and published in 2024 in Dolomites Res. Notes Approx.

The content of Chapter 3 is also a part of the above mentioned paper; but the version in thesis has more details and explains more proofs. This chapter is devoted to a method of constructing new pseudo-Leja sequences by using the construction called intertwining. The most important facts proved here is Theorem 3.2.1, which (under appropriate assumptions) gives a natural algorithm- how to construct, using intertwining procedure, pseudo- Leja sequences in $\mathbb{C}^{n_1+n_2} = \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$, corresponding to the set $K := K_1 \times K_2$.

This is a generalization of earlier result of Irigoyen, who proved the theorem for products of (complex) one- dimensional compact sets and Leja sequences. The main idea is the same- to use the intertwining construction. Certainly, this part of Thesis uses also the results of Calvi (reference [37]).

Chapter 3.3 of the dissertation contains a result analogous to previous results of Siciak (reference [91]) and Białas-Cież and Calvi (reference [11]). So, having two compact, regular, polynomially convex subsets of $\mathbb{C}$, and two sequences of points $\{a_j\} \subset K_1$, $\{b_j\} \subset K_2$, satisfying the natural condition (3.22), one produces intertwining sequence, and one proves (see Theorem 3.3.3) that this is a good set of nodes for polynomial approximation, i.e., for any complex valued function f defined in a neighbourhood of $K_1 \times K_2$ the interpolating polynomials converge uniformly to F on some compact neighbourhood of $K_1 \times K_2$.

The result is a nice combination of previous results ([91] and [11]). The assumptions, formally, are slightly weaker than in the above mentioned papers. The proof (non- trivial) follows the approach of [91]; the author also discussed the issue of identity theorem for pseudo- Leja sequences (p.27 l 1-3) and clarifies it.

Chapter 4 introduces the reader into computational applications of the previously introduced objects. The algoriths producing "discrete Leja points" are described (and supported by Propositions 4.1.1 and 4.1.2). In particular, in Sections 4.2.1 and 4.2.2 the author presents the results of numerical experiments, which, according to the theoretical arguments, should give (in a limit) the equilibrium measure in the ball

$B \subset \mathbb{R}^2 \subset \mathbb{C}^2$ and the logaritmic capacity of the unit circle in $\mathbb{C}$. Since both are well known, one can observe the convergence to the limit and the speed of convergence.

Finally, in the third, most elaborated example, the author describes the numerical experiment for the function given by the formula (4.6), and for its interpolating polynomials with interpolation nodes being pseud Leja points. The compact set $K \subset \mathbb{C}^2$ is either a square $[-1, 1] \subset \mathbb{R}^2 \subset \mathbb{C}^2$ or a unit ball in $\mathbb{R}^2$. A calculation performed and explained in detail (4.8)- (4.22) shows the parameters for which the interpolating polynomials should converge fast to the initial function and, conversely, the parameters for which this convergence should rather fail. All this is confirmed by numerical experiments.

Chapter 5 follows closely the content of the paper of the candidate written jointly with L. Białas- Cież, A. Sommariva nad Vianello.

The main theoretical results are Proposition 5.2.2 and Proposition 5.2.5. The authors show how to construct optimal admissible meshes of Chebyshev type for sufficiently regular complex curves or domains.

Proposition 5.2.3 gives then a precise estimate from above and from below of the Lebesgue constant (norm of the interpolation operator).

The chapter is completed by a series of numerical tests, well ilustrating the described phenomenon.

Chapter 6, (also based on a joint paper with the same three coauthors) is devoted to similar questions: Proposition 6.2.2 provides a precise estimate for the Lebesgue constant of the interpolation operator, on $K = [-1, 1]^n$, and a product Chebyshev mesh. The optimal admissible polynomial mesh on a simplex is produced through reduction to the previous case of $K = [-1, 1]^n$. A natural parametrization of n dimensional ball by spherical change of variables, allows also to obtain analogous results for a ball in $\mathbb{R}^n$. The chapter is illustrated by a series of numerical experiments, described in detail.

The last Chapter 7 is of different character, and it provides an improved estimate from below of the transfinite diameter of a Bernstein set. This is a very classical question, many estimates have been proved for Markov and Bernstein sets. So, it is quite surprising, that such a natural strengthening was possible. This new estimate is provided in Theorem 7.1.2. It improves the previously known estimate significantly, in all dimensions $n > 1$. The proof is not long, but nice and effective. The author uses simply the methods elaborated in previous sections: Leja sequence allows to express the transfinite diameter in terms of (limit of) corresponding Vandermonde determinants. And these are estimated from below using Bernstein (Markov) property.

Conclusion

My opinion on the thesis is definitely positive. The author works in a modern, active area of theoretical mathematics and its numerical counterpart. The dissertation contains a series of results obtained by the candidate himself (with no coauthors), as well as another series of results being obtained as a fruit of good collaboration.

The author made a nice contribution to the theory of polynomial approximation. Especially, I would like to underline the content of Chapter 7 and Chapter 2, as well as a very detailed discussion on modification of Siciak's proof, performed in Chapter 3.

The thesis is well written, with a good introductory chapter, completed by rich and carefully prepared bibliography. The author uses succesfully a bunch of advanced tools and methods, also coming from the (classical now) papers listed in the bibliography. I have some remarks and comments concerning the content of Thesis (listed below); they are of rather editorial character and do not affect my positive opinion.

I am convinced that the thesis presented by Dimitri Jordan Kenne fulfil the requirements for gaining the PhD degree.

Podpisany elektronicznie przez
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Additional specific comments and remarks

The double index in (1.1) looks strange at the first reading. Perhaps the author would like to say that he considers a trangle array?

p. 4 It would be good to write a separate definition of the unisolvent set, and to refer to it at p.5 l.3 (i.e. to say that the determinant is non- zero, and thus the existence is guaranteed. Moreover, what the author means by saying that the uniqueness is guaranteed? $L_{A_j}e_j$ is given by the formula (1.3).

p.5 l-2 The sentence *The sufficiency follows from an application of (1.4)* lacks sufficient detail.

p.6 l -3. Here, the author turns from $\mathbb{C}^n$ to $\mathbb{C}$, without mentioning this.

p. 9 Proposition 1.3.1 I understand that here, the sequence $L_d f$ depends on the array $\{\xi_{dj}\}$. If so, this should be said explicitly.

p. 10 l -1 is it *Carlson's theorem* or *Carleson's theorem*?

p. 11 (1.36) is not completely evident at the first reading.

p.16 l 1 $\mathcal{P}_N(\mathbb{C}^n)$ denotes the space of polynomials of degree at most N? If so, its dimension is much larger than N.

p.17 formulation of Proposition 2.1.1 is not strict (*always possible*- what does it mean?)

p. 19-20 proof of Theorem 2.2.1. It would be natural (and required!) to compare the proof with the proof in [11]. In particular, Theorem 2 in that paper.

p. 23 Theorem 3.1.2 is similar to Lemma 1.1.1. I think that this similarity should be mentioned and commented.

p.26 a comment on Theorem 3.2.1 - as a generalization of Irigoyen's result. Is the proof presented in the thesis similar to that in [60]? What new difficulties appear when passing fro $n_1 = n_2 = 1$ to higher dimension. Also, it would be good to mention the results of [31] in this context.

p.27 l-11 there is no Definition 1.24 in the text. Does the author mean formula (1.24)?

p.28 l -7 It is not sufficently explained how this formula follows from (3.22) and Remark 1.

p.30 last sentence of the proof : *(...) we can apply(...) twice in row.* Could you explain more precisely the meaning of this sentence?

p. 31 Formulation of Proposition 4.1.1. I had trouble with quantifiers in the formulation of this proposition. So, first $d \in \mathbb{N}$ is given, and a set A which is determining for the space of polynomials of degree $\leq d$, and then in item 2. one computes the points for all $d \in \mathbb{N}$? Perhaps it should be : for every $d' \leq d$?

p. 34 perhaps it would be good to say more about the admissible meshes used in Figures 4.3 and 4.4 (and coming from the paper [32]).

p.36 Here, the Author turns to the example described in [5]; it would be easier for the reader if the Author writes explicity, that the ball is understood as a subset of $\mathbb{B}^2$, naturally embedded into $\mathbb{C}^2$. and also the equllibrium measure is understood in this sense. I needed to open the paper [5] to realize about the setting.

p.38 the first line after the algorithm. There is no Theorem 8.1 in the paper [91]. Moreover, is the variable z in $\mathbb{C}$ (as it is written in the paper) or rather in $\mathbb{C}^2$?

Examples displayed at Figures 4.7 and 4.8. I did not find the information what collection of pseudo Leja points was taken for these examples.

Remark about Chapter 5 of the Thesis. This Chapter is based on the paper [13] written by four authors, including the PhD candidate and his advisor. Of course, including such joint papers is perfectly correct. However, it woud be good if the author of Thesis commented about his contribution in this collaboration.

p. 43 l -4 Which orthonormal polynomials are meant in this paragraph? Some specially chosen ones?

p. 45, the third line of the proof of Proposition 5.2.2. Here, the author refers to Lemma 1. However, there is no Lemma 1 in this text.

l. 45, formulation of Proposition 5.2.3. Shouldn't it refer rather to Lemma 5.1.1?

p. 54 formula (6.15). The meaning of the notations $\sim$ and $\approx$ is not explained.

p.56 formula (6.22): shouldn't it be λ_d instead of λ_n?

p. 56 formula (6.21) what is the meaning of the constant c which appear here?

p.68 l -8 It would be good to write how Lemma 7.2.1 guaratees the formula in l–7.

p.69 Remark 14. The authors explains one implication (an immediate one), the other one is said to be straightforward. Nevertheless, it would be good to explain also the second implication.

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