Report on the PhD thesis ”Classification problems in topological dynamics and ergodic theory” by Konrad Deka.

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This thesis is situated at the intersection of descriptive set theory, ergodic theory, and symbolic dynamics. It is structured into three main parts. The first part is dedicated to studying the Borel complexity of sets of interest in ergodic theory. The second part examines the complexity of the isomorphism relation for certain subshifts. The third part aims to demonstrate that the isomorphism and flip conjugacy relations between minimal Cantor systems are not Borel. In addition to these core sections, the thesis includes an introduction and a preliminaries section, both of which are clear and self-contained. In the following, I will describe the content of the thesis in more detail, concluding with some final observations.

Chapter 3 studies the Borel complexity of generic sets of the invariant probability measures of dynamical systems (X, T), where X is a compact metric space and $T : X \to X$ is continuous. More precisely, given a dynamical system (X, T) and an invariant probability measure μ of (X, T), the goal is to study the Borel complexity of the set

$$\mathrm{Gen}(\mu) = \left\{ x \in X : \mu = \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \delta_{T^j x} \right\}.$$

In Proposition 3.3.1, it is shown that $\mathrm{Gen}(\mu)$ is always Π^0_3. Thus, the problem is reduced to determine if it is possible to realize the lower complexities and Π^0_3-complete generic sets. The first result in this direction is presented in Section 3.4, where Deka shows that it is possible to realize the Π^3_0-complete complexity within the generic sets of Toeplitz subshifts (one and two-sided). More precisely, he shows that the subshifts $X \subseteq \{0, 1\}^{\mathbb{N}}$ introduced by Oxtoby in 1952 have exactly two ergodic measures μ_0 and μ_1 that satisfy $\mu_0([1]) = a < \mu_1([1]) = b$. Then, using the combinatorics of the elements in X, he defines a map f from $\mathbb{N}^{\mathbb{N}}$ to X, showing that f is a continuous reduction from a Π^0_3 set of $\mathbb{N}^{\mathbb{N}}$ to $\mathrm{Gen}(\mu_0)$. Consequently, he gets Theorem 3.4.5 and Corollary 3.4.6 about the realization of Π^3_0-complete generic sets. Then, in Section 3.5, Deka constructs a minimal subshift $X \subseteq \{0, 1\}^{\mathbb{Z}}$ having an ergodic measure μ such that $\mathrm{Gen}(\mu)$ is equal to the intersection of a F_σ set and a G_δ, from which follows that $\mathrm{Gen}(\mu)$ is contained in $\triangle^0_3$ and not below. This subshift X is defined using the block concatenation method introduced in Chapter 2, similar to the S-adic construction of subshifts. In Section 3.6, using several examples of the literature, Deka shows that within the family of transitive dynamical systems, it is possible to realize strictly all the Borel complexities below Π_3^0, except Σ_1^0, which is a consequence of Corollary 3.6.2.

Chapter 4 requires the following concepts: Let X and Y be two polish spaces (separable completely metrizable), endowed with equivalence relations E and F, respectively. A *Borel reduction from E to F*, is a Borel map $f_X \to Y$ such that xEx' if and only if $f(x)Ff(x')$. In this case, we say that E is Borel reducible to F and is written as $E \leq_B F$. In Chapter 4, using the order relation $\geq_B$, Deka classifies equivalence relations induced by the isomorphism relation on the set of subshifts in $A^{\mathbb{Z}}$, where A is some fixed finite alphabet. On $\mathcal{K}(A^{\mathbb{Z}})$, the space of subshifts of $A^{\mathbb{Z}}$ equipped with the topology of the Hausdorff metric, conjugacy between subshifts induces a natural equivalence relation on $\mathcal{K}(A^{\mathbb{Z}})$. The restriction of this relation to the class of subshifts with specification is denoted by $\cong_{S_p}$. The main result of this chapter states that $\cong_{S_p}$ is not amenable, which implies that subshifts with specifications cannot be classified in a computable Borel way, using a numerical invariant (as the entropy, in the case of the Bernoulli shifts). For this, he used some already known results concerning the amenability of equivalence relations.

Finally, Chapter 5, is devoted to the study of $\cong_{min}$, the restriction of the isomorphism relation to the minimal homeomorphisms in Homeo($\mathcal{C}$), where $\mathcal{C}$ is the Cantor set. The goal is to show that $\cong_{min}$ is not Borel, answering a question formulated by Gao in 2022. The general idea is to find a Borel map $\Phi : Trees :\to \mathrm{Min}(\mathcal{C}) \times \mathrm{Min}(C)$, where $Trees$ is the space of trees with infinite levels, and Min($\mathcal{C}$) is the space of minimal homeomorphism on $\mathcal{C}$, such that Φ is a reduction from IF (the elements of $Trees$ with at least one infinite path) to $\cong_{min}$. Then, using the fact that IF is not Borel, it is proven that $\cong_{min}$ is not Borel. Consequently, he also obtains that the flip conjugacy relation is not Borel. For the construction of the map Φ, several tools are introduced, such as the family of construction sequences (subshifts defined by special block concatenations) and the inverse systems of groups associated with a tree, which is used to associate it with a system of constructions sequences, which in turn defines an inverse system of subshifts with the required properties.

I particularly appreciated the construction of concrete examples in Section 3. I wonder whether it will be possible to establish Borel realization results using only Toeplitz subshifts or some other specific family of subshifts. I am also curious about what happens to the complexity of generic sets for measure spaces with more complicated geometries or whether it might be possible to tackle such problems using Bratteli-Vershik diagrams. I thoroughly enjoyed reading Section 4, as the proofs are seemingly simple yet profound. The technical level of the final chapter is quite impressive. At this point, I wonder if these techniques could be used to compare actions that do not come from the same group. For example, could they provide insight into the long-standing open question of whether every minimal free continuous action of an amenable group on the Cantor set is orbit equivalent to one with the same properties of $\mathbb{Z}$?

The thesis is very well-written, employing techniques from various areas of dynamical systems theory. It presents original results that are both deep and highly relevant while also raising intriguing questions for further exploration. As a result of this work, at least two publications have been submitted. The overall level of the thesis is of exceptionally high quality.
I propose to award the dissertation of Mr. Konrad Deka the distinction (cum laude).

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