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To whom it may concern:

The purpose of this letter is to provide my evaluation of Mr. Konrad Deka's PhD thesis, titled "Classification Problems in Topological Dynamics and Ergodic Theory." In my opinion, as I will elaborate in the following, this thesis offers *original solutions to at least three significant scientific problems*. The complexity of these problems, along with the range and depth of the methods employed, highlight the *exceptional high quality of the work*. The dissertation meets the rigorous standards expected of a PhD thesis, and I strongly recommend its acceptance with *distinction.*

Mr. Deka's research centers on the theory of topological dynamical systems within the classical framework. In this context, the focus is on the long-term iterative behavior of a homeomorphism T on a compact metric space X, with the goal of identifying topological and statistical dynamical invariants associated with T. The pair (X, T) is referred to as a topological dynamical system.

The development of topological dynamics frequently parallels that of ergodic theory, which studies measure-preserving transformations on standard Lebesgue spaces. These two fields are deeply interconnected, often complementing and influencing each other in significant ways. One of the most fundamental questions in the theory of dynamical systems is the classification of systems up to conjugacy or flip-conjugacy, where a dynamical system (X, T) is conjugate to $(Y, S^{\pm 1})$, meaning they are structurally equivalent up to time reversal. This problem has been central to the field for over 90 years, dating back to the foundational works of Koopman and von Neumann on the spectral theory of dynamical systems. Since then, numerous invariants of conjugacy have been proposed in an attempt to identify a complete set of invariants capable of fully classifying the conjugacy classes of dynamical systems. While certain invariants, such as Kolmogorov's entropy and specific spectral properties, have proven effective in classifying particular classes of measure-preserving systems, the quest for a complete set of invariants has remained an open and challenging problem for many decades.

A significant breakthrough occurred in the early 2000s with a series of papers, most notably by Hjorth (2001) and later by Foreman, Rudolph, and Weiss (2011). These works demonstrated that no reasonable classification of measure-preserving systems by a countable protocol is possible. Specifically, they established that, in the context of ergodic theory, the isomorphism equivalence relation on ergodic transformations is complete analytic and thus not Borel, highlighting the intrinsic complexity of the isomorphism relation for measure-preserving systems. This shift in focus toward the impossibility of classification, or the complexity of classification problems in dynamical systems was largely driven by the introduction of descriptive set-theoretic methods. These methods, pioneered by researchers such as Kechris, Weiss, Miller, Gao, Jackson, Dougherty, Louveau, and others, have profoundly influenced the study of ergodic theory, Borel dynamics, and more recently, topological dynamics. The investigation of set-theoretic complexity represents a contemporary and powerful approach to understanding the classification of diverse mathematical structures.

The overarching theme of Mr. Deka's thesis is the study of the set-theoretic complexity of classification problems in topological dynamics, focusing on three primary areas: (1) the study of Borel complexity of generic points for a given invariant measure; (2) the study of the descriptive set-theoretic complexity of the isomorphism problem for subshifts with specification property; and (3) the study of the descriptive set-theoretic structure of the isomorphism relation on Cantor minimal systems.

(1) Given a topological dynamical system (X, T) and a T-invariant measure μ, a point $x_0 \in X$ is called μ-generic if the sequence of measures $\{(\delta_{x_0} + \delta_{Tx_0} + \cdots + \delta_{T^{n-1}x_0})/n\}$ converges to μ. In the first part of the thesis, Mr. Deka studies the Borel complexity of the set of generic points $\mathrm{Gen}(\mu)$ of μ and its dependence on dynamical properties of the system (X, T). The set $\mathrm{Gen}(\mu)$ is always Borel and belongs to the class Π^0_3. Mr. Deka investigates whether it is possible to construct a dynamical system whose set of generic points has lower Borel complexity than Π^0_3, such as G_δ, F_σ, etc.

Using classical techniques from symbolic dynamics, including the construction of Toeplitz systems with predefined characteristics, he provides explicit examples of two minimal systems: one where the set of generic points is Π^0_3-complete, and another where it is of class Δ^0_3. Furthermore, he demonstrates that sets of generic points with any Borel complexity strictly lower than Π^0_3 can be realized within the class of transitive systems. Results of this chapter have appeared in a joint paper of the author with Jackson, Kwietniak, and Mance.

(2) The second part of the thesis addresses a problem posed by Rufus Bowen in his famous notebook, concerning the classification of subshifts with the specification property—commonly referred to as Problem 32. One of the chapters in Mr. Deka's thesis is devoted to the study of the descriptive set-theoretic complexity of the isomorphism relation for symbolic subshifts with the specification property.

The natural framework for comparing equivalence relations on Borel spaces is provided by Borel reducibility, a concept that goes back to the seminal work of Friedman and Stanley (1989). Equivalence relations that admit real numbers as complete invariants constitute the simplest level in this hierarchy. The next level is occupied by hyperfinite equivalence relations, which are characterized as those induced by Borel actions of the group of integers $\mathbb{Z}$. At the highest level of complexity within the realm of countable Borel equivalence relations lies the universal countable Borel equivalence relation, which arises from the natural action of the free group $\mathbb{F}_2$ on the space $\{0, 1\}^{\mathbb{F}_2}$. This relation is maximal in the quasi-order of Borel reducibility, serving as a benchmark for the most complex classification problems within the Borel framework.

In this thesis, Mr. Deka provides a lower bound on where the isomorphism relation for subshifts with the specification property resides within this hierarchy. Namely, he shows that the isomorphism equivalence relation is not amenable. In particular, it is not hyperfinite, which provides a negative resolution to Bowen's original question, in the sense that it rules out the possibility of a classification by concrete invariants.

In his proof, Mr. Deka considers the space of all subshifts over a five-letter alphabet and constructs a non-amenable group G (containing $\mathbb{F}_2$) of automorphisms acting on this space. He demonstrates that the isomorphism relation for subshifts with the specification property contains a sub-equivalence relation given by the orbit equivalence relation induced by the action of G, which is shown to be non-amenable. I found the proof to be very elegant, yet technically sophisticated. The results of this chapter have already appeared on ArXiv as a joint work with Kwietniak, Peng, and Sabok.

(3) In "Open Questions in Descriptive Set Theory and Dynamical Systems" (2023), Su Gao raised the question of whether the isomorphism relation on Cantor minimal systems is Borel—a crucial problem in the effort to understand the classification of Cantor dynamical systems. In Corollary 5.7.2 of his thesis, Mr. Deka establishes that the conjugacy relation and the flip-conjugacy relation are complete analytic and therefore not Borel. This result demonstrates that Cantor minimal systems are not classifiable by countable protocols.

In my opinion, this is the most significant result of the thesis and one that merits publication in top-tier mathematical journals. This single result alone would justify awarding a PhD with distinction. A big picture idea behind the proof is the construction of a Borel function f from the set of all trees into the space of pairs $\mathcal{T} \mapsto f(\mathcal{T}) = ((X_1, T_1), (X_2, T_2))$, where each component is a Cantor minimal system. This function f has the property that (X_1, T_1) and (X_2, T_2) are topologically conjugate if and only if the pre-image tree $\mathcal{T}$ is ill-formed. Using the fact the set of ill-formed trees is not Borel, the author establishes the result. Overall, the proof is highly nontrivial and relies on a brilliant adaptation of techniques from Foreman, Rudolph, and Weiss (2011).

One of the aspects that could be strengthened in the thesis is the historical account and literature review, which is somewhat limited. This may make it more challenging for non-experts to fully place the results within the broader context of dynamical systems theory, appreciate their significance, and understand the direction in which the field is evolving.

Overall, Mr. Deka has demonstrated a deep understanding of the methods in topological and symbolic dynamics, ergodic theory, and the aspects of descriptive set theory relevant to the study of Borel equivalence relations and their classification. His research is both novel and impactful to the field. I strongly believe that the results obtained by Mr. Deka are of exceptional quality, and the thesis merits recognition with *distinction*.

Sincerely,

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