Report on the PhD thesis of Konrad Deka
by
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Konrad Deka's thesis concerns the descriptive theory of sets coming from dynamical purposes. Most of the results are about Borel complexities of the set of generic points with respect to some measure and of the isomorphism relation.

There are some general results but the main part of this thesis gives the existence of some "dynamical sets" having specific Borel complexities.

1. Chapter 1 : Introduction

The introduction is short. It briefly describes the chapter and gives some elements of context. It presents the three main questions of this thesis :

- Chapter 3 : What is the Borel complexity of the set generic points relatively to an invariant measure ? It focuses on the constraints that minimality of topological dynamical systems impose to the Borel complexity of the sets of generic points.
- Chapter 4 : What are the properties of the isomorphism relation between dynamical systems within the framework of Borel reducibility of equivalence relations ?
- Chapter 5 : What is the complexity of the isomorphism and flip isomorphism between minimal Cantor systems within the framework of Borel reducibility of equivalence relations ?

2. Chapter 2 : Preliminaries

In this section, the classical definitions of topological dynamics and combinatorics on subshifts are given. A part is devoted to the basics on invariant measure. Two classical families of dynamical systems are defined namely odometers and Toeplitz subshifts.

It is introduced a classical way to build by concatenation examples of subshifts. This method will be useful later (chapters 3 and 5) in the thesis to construct specific examples.

3. Chapter 3 : Borel complexity of the set of generic points for a measure

A result of Sharkovsky and Sivak (2016) proved that if μ is an invariant measure of a topological dynamical system (X, T) then the set of its generic points $\mathrm{Gen}(\mu)$ is Π^0_3. In the same paper they ask whether minimality of dynamical systems put any constraints on the Borel complexity of this set. This is the framework of this chapter.

A construction given by Oxtoby (1952), that defined what was later called Toeplitz subshift, provided an example of a subshift having two ergodic measures and thus infinitely many invariant measures. Konrad inspected this construction to show that one of the two ergodic measures of this Toeplitz subshift has its set of generic points which is Π^0_3-complete, that is Π^0_3 but not Σ^0_3.

To go further to find examples that are Δ_3^0 but not of lower Borel complexity needs much more attention. This is what Konrad provides in Section 3.5 (Theorem 3.5.1) : there exists a minimal subshift having an ergodic measure μ such that $\mathrm{Gen}(\mu)$ is Δ_3^0 but not of lower complexity.

This is a technical section where along the description of the inductive block construction of the subshift, properties of words involved are given. In particular they should have a property, called "uniform", on the discrepancy of letters inside these words. This is a key notion that is used to select an ergodic measure μ such that $\mathrm{Gen}(\mu)$ is Δ_3^0 and not of lower complexity.

In Section 3.5.1 is proven that the resulting subshift is minimal. More important, the ergodic measure μ is selected to maximize the frequency of some fixed letter. Thanks to the uniform property of the construction, this defines the desired ergodic measure.

In Section 3.6 Konrad investigates the possible Borel complexity of $\mathrm{Gen}(\mu)$ through examples from the litterature.

In Section 3.7 he considers non minimal dynamical systems, more precisely systems having "specification properties", like mixing subshifts of finite type, and shows, for example that for any invariant measure of these systems $\mathrm{Gen}(\mu)$ is Π_0^3-complete.

4. Chapter 4 : Equivalence relations originating from dynamical systems

In this chapter the Borel complexity of the isomorphism relation between subshifts over a given alphabet is studied. Konrad recall that Sabok and Tsankov (2017) showed that the isomorphism relation inside a class of Toeplitz subshifts is amenable.

Konrad first shows (Theorem 4.3.3) that, for the whole class of transitive systems on a 5-letter alphabet, the relation is not amenable. In Theorem 4.3.4 this is proven for the subclass of subshifts with the specification property. As a comment Konrad added that this is also true for larger alphabets. This is still true within the family of pointed transitive subshifts on $\{0, 1\}$.

5. Chapter 5 : The complexity of isomorphism relation of minimal Cantor systems

Chapter 5 in a continuation of Chapter 4 where Konrad focuses on minimal Cantor systems. It is shown that the isomorphism relation for this family is not amenable. The proof consists of a long and technical preparation of lemmas involving many tools like block concatenation subshifts, inverse limits and trees.

6. Conclusion

The results presented in this thesis are interesting and highly non trivial. They complement the already existing results on the Borel complexity of sets of dynamical origins. In proving his results, Konrad demonstrated a great expertise to work with complex constructions in order to highlight de Borel complexity of dynamical behaviours. They are at a level I associate to a PhD.

There's no doubt that the results obtained by Konrad deserves the title of doctor. In conclusion, I give a favorable opinion for Konrad's doctoral defense to obtain the degree of Doctor.