Review on Ph.D. thesis
"Some applications of graded Kac–Moody Lie algebras"
submitted by Tymoteusz Chmiel

Reviewer: Kyu-Hwan Lee (University of Connecticut)

This thesis is concerned with applications of graded Kac–Moody Lie algebras across three subjects: (1) investigating certain classes of Koszul modules and resonance varieties, (2) constructing equivariant embeddings of homogeneous spaces, and (3) studying the structure of Gorenstein ideals of codimension four. Kac–Moody algebras are infinite dimensional analogues of finite dimensional simple Lie algebras. Extensions of classical constructions involving finite dimensional simple Lie algebras often lead to intriguing questions related to Kac–Moody algebras. The three subjects considered in this thesis are both important and challenging, and the results are valuable and significant.

The results on the first subject are characterization of nilpotent Kac–Moody Koszul modules by establishing a simple condition on generalized Dynkin diagrams, precise description of all finite-length Kac–Moody Koszul modules, and computation of equivariant resonance varieties for types G_2, F_4, E_6, and E_7 along with homological invariants of the corresponding Koszul modules. Generic Koszul module are also studied.

As the second subject of the thesis, the author considers embeddings of homogeneous spaces arising from Kac–Moody groups into Grassmannians of irreducible representations. More precisely, he establishes the existence of an embedding $G/P \hookrightarrow \mathrm{Grass}(d, V(\lambda))$ for a particular integer d, where P denotes a maximal parabolic subgroup of a Kac–Moody group G and $V(\lambda)$ is an irreducible representation of G with highest weight λ. The cases of simply-laced Dynkin diagrams of types A_n, D_n and E_6 are thoroughly analyzed.

The final subject of the thesis deals with minimal free resolutions of Gorenstein ideals of codimension four. Here, the higher structure maps appear in graded components of critical representations, and the author investigates higher structure maps of ideals with six generators and obtains a proof of the conjecture that every Goren-

stein ideal of codimension four with six generators can be realized as a hypersurface section of a Gorenstein ideal of codimension three.

As I am most familiar with the second subject, I will elaborate on the author's results on this problem. When G is a simple algebraic group and P is a parabolic subgroup of G, the quotient space G/P has a structure of projective variety. In particular, when P is a maximal parabolic subgroup, we have an embedding $G/P \hookrightarrow \mathbb{P}(V(\omega_\alpha))$ where ω_α is the fundamental weight corresponding to P and $V(\omega_\alpha)$ is the irreducible representation of G with highest weight ω_α.

The main result on the second subject is a generalization of the standard embedding described above using some other irreducible representations, which can also be extended to Kac–Moody groups. This problem is challenging not only because it requires the consideration of more complicated representation theory but also because it involves handling infinite-dimensional objects, such as ind-varieties. The author overcomes all technical difficulties and obtains quite a satisfactory generalization of the standard embedding in Theorem 4.1.46. Moreover, the author provides all the details for classical types A_n, D_n and exceptional type E_6 in Sections 4.2 and 4.3, respectively. These examples provide valuable computational results for researchers in this field.

My assessment of the quality of this thesis can be summarized as follows:

1. Originality and Innovation. The thesis showcases an appropriate level of originality and innovation, addressing a complex and challenging problem with a fresh perspective. The author has clearly identified important questions in existing research and developed solutions to advance the knowledge frontier in the research domain. This work is a valuable addition to the field.

2. Methodological Rigor. The research methodology employed in this thesis is thorough and meticulously executed. The author has demonstrated a sound understanding of advanced concepts and has applied them with precision. The use of advanced techniques in proofs and computations reflect a high level of competence.

3. Comprehensive Understanding of Existing Results. The thesis is based on an extensive and well-organized literature review, demonstrating that the author has a

comprehensive understanding of the current state of research in the field. The author situates his thesis within the broader context of existing work, highlighting both the strengths and limitations of previous studies. This sets a solid foundation for the research questions and hypotheses, underscoring the significance and relevance of the author's contributions.

4. Clarity and Structure. This thesis is very well organized and clearly written. The first two chapters establish a solid foundation, while each of the subsequent three chapters addresses one of the main subjects in depth. The clarity of expression and the careful organization of ideas are commendable. However, there are quite a few typographical errors, and I recommend that the author conduct a thorough revision before the final submission.

5. Contribution to Knowledge. The findings of this thesis represent a substantial contribution to the field. The thesis tackles complex theoretical concepts with a high degree of sophistication. The author demonstrates an ability to engage critically with these ideas, overcoming obstacles with both persistence and creativity.

In conclusion, this Ph.D. thesis meets and exceeds the standards expected for a Ph.D.-level work. It demonstrates a commendable depth of understanding, rigorous research methodology, and significant contributions to its field. Therefore, it deserves to be accepted with recognition for its quality, originality, and contribution to advancing knowledge.

I attest that this thesis constitutes an original solution to a scientific problem in compliance with the requirement of Article 187, Section 2 of the Law of Republic of Poland on Higher Education and Science of July 20, 2018.

Kyu-Hwan Lee, Ph.D.

Professor of Mathematics
University of Connecticut