Thesis evaluation – "Some applications of graded Kac-Moody Lie algebras", by Tymoteusz Chmiel

Reviewer: Claudiu Raicu

1 Overview

Kac–Moody Lie algebras constitute a class of infinite-dimensional Lie algebras that generalize finite-dimensional semisimple Lie algebras. Named after Victor Kac and Robert Moody, who independently developed them in the 1960s and 1970s, Kac–Moody Lie algebras arise as an extension of the Cartan classification of Lie algebras and play a crucial role in various areas of mathematics and theoretical physics.

The thesis under review establishes a variety of novel applications of Kac–Moody algebras within the fields of commutative algebra and algebraic geometry. The thesis is well-written, and discusses a variety of interesting problems, results, and conjectures. It contains novel ideas and techniques, and it is a valuable addition for researchers working at the interface of representation theory, commutative algebra, and algebraic geometry.

Organization of the thesis:

- The introduction outlines the context and main results of the thesis.
- Chapter 1 summarizes the basic theory of Kac–Moody Lie algebras and groups, as they will be used in subsequent chapters.
- Chapter 2 discusses some of the technical background related to the three directions explored in this thesis: Koszul modules, homogeneous spaces, and free resolutions.
- Chapter 3 is concerned with the study of Koszul modules arising in the context of Kac–Moody Lie algebras, as well as their resonance varieties and related homological invariants.
- Chapter 4 examines embeddings into classical Grassmannian varieties for homogeneous spaces of Kac–Moody groups.

- Chapter 5 analyzes Gorenstein ideals of codimension four, and discusses the proof of a folklore conjecture for ideals with 6 generators.
- The thesis also includes an Appendix containing extensive data on the resonance and homological invariants of Kac–Moody Koszul modules, as well as computer algebra code describing the spinor coordinates for the resolution of a Gorenstein ideal of codimension 4.

2 Koszul modules

Koszul modules are finitely generated graded modules over a polynomial ring, that have been studied prominently by topologists in the form of *Alexander invariants of groups* [PS04, PS06, SS06, DPS09, PS12, PS15]. More recently they have been shown to be fundamental objects in algebraic geometry and commutative algebra, bearing numerous analogies with Koszul cohomology groups. There are several recent applications of Koszul modules in the literature: to the study of syzygies of curves [AFP$^+$19], to invariants of Kähler groups or hyperplane arrangement groups (such as the Chen ranks, degree of growth, or virtual nilpotency class) [AFP$^+$22], to stabilization results for the cohomology of thickenings, ramification divisors, canonical pencils, resonance divisors [AFRW24], etc.

The terminology *Koszul modules* first appears in [PS15], where the authors study the vanishing of resonance varieties associated to representations of semisimple algebras. Chapter 3 of the thesis under review proposes to go beyond the Papadima–Suciu investigation and study Koszul modules associated to representations of Kac–Moody Lie algebras. In this context, Theorem 3.1.31 presents a characterization of finite lenght Koszul modules, which parallels the main result of [PS15] in the classical Lie algebra setting. By combining Theorem 3.1.31 with the classification of representations with finitely many orbits, one gets a classification of the Kac–Moody Koszul modules that do not have finite length, in which case the natural geometric invariant to study is the support of the Koszul module, that is, the so-called *resonance variety*. This involves a case-by-case analysis, and the description of the resonance for infinite length modules is given in the Appendix.

The interest in Kac–Moody Koszul modules comes from the fact that they give an interesting and explicit class of examples, and one can hope to extrapolate properties observed in these special cases to the general setting. Section 3.3 of the thesis discusses perhaps the most fundamental open question regarding homological invariants of Koszul modules, which is to find an effective absolute bound on their Castelnuovo–Mumford regularity. In the case of finite length modules, this goes back to a question left open by [PS15], which was resolved subsequently in [AFP$^+$19] – the infinite length case is significantly more challenging. One standard interpretation of Castelnuovo–Mumford regularity is as a bound for the height of the Betti table of a module. Theorems 3.3.41 and 3.3.42 provide the first general results that I am aware of regarding explicit descriptions of Betti tables for Koszul modules.

3 Grassmannian embeddings

A highly effective method for studying abstract varieties is by considering embeddings into special varieties, and taking advantage of the geometry of the ambient space in order to shed light on the properties of the original variety. The most common embeddings are those where the ambient space is a projective space, and for instance in the case of curves this (together with the study of more general maps to projective space) is the subject of Brill–Noether theory, one of the great achievements in the study of algebraic curves. More general embeddings include embeddings into Grassmannians and flag varieties, toroidal embeddings, spherical embeddings, etc. For abstract varieties with a large group of symmetries, it is natural to consider embeddings that are equivariant with respect to the group action, and this is the context considered in Chapter 4 of the thesis under review.

In the context of studying resolutions of perfect codimension three ideals and the associated generic rings [Wey18], Weyman has discovered a surprising action of the Kac–Moody Lie algebra associated with a Dynkin diagram of type $T_{p,q,r}$. A choice of a node of the Dynkin diagram has both algebraic and geometric interpretations: on the one hand, it gives rise to a fundamental representation, and on the other hand it produces a maximal parabolic subgroup of a Kac–Moody group and an associated homogeneous space. In his work on $T_{p,q,r}$ diagrams, Weyman observed that under suitable finiteness assumptions it is often possible embed the Kac–Moody homogeneous space associated to one node, into a Grassmannian associated to the fundamental representation of another node.

Chapter 4 of the thesis under review takes a more systematic look at Weyman's construction, and provides in Theorem 4.1.46 a more general context when such embeddings exist. The generalization goes in two different directions: on the one hand it is not restricted to $T_{p,q,r}$ diagrams, and on the other hand it doesn't only consider Grassmannians of fundamental representations. The construction of the embeddings is done in abstract terms in the proof of the theorem, but the author also gives a more concrete realization subsequently.

4 Gorenstein ideals of codimension 4

Gorenstein ideals are an important class of ideals arising in commutative algebra and algebraic geometry, specifically in the context of duality theory. They have applications in singularity theory and linkage theory, and occur in the study of canonical curves, K3 surfaces, and more general Calabi-Yau varieties. Understanding the structure of Gorenstein ideals is then a fundamental problem, with a wide range of applications.

All hypersurface rings are Gorenstein, and in codimension two the Gorenstein and complete intersection properties are equivalent by a result due to Serre [Ser63]. The biggest success in the structure theory of Gorenstein ideals comes from the celebrated Buchsbaum–Eisenbud theorem [BE77], which provides a very useful description of Gorenstein ideals of codimension 3. Specifically, it states that any Gorenstein ideal of codimension 3 can be realized as the Pfaffian ideal of a skew-symmetric matrix. Extending this result beyond codimension 3 has proved to be a challenging task [KM80, Rei15, CLW23].

The main ingredients in the study of codimension 4 Gorenstein ideals are the multiplicative structure on their free resolution [KM80], and the Buchsbaum–Eisenbud multiplier maps (arising from the *First Structure Theorem* [BE74, Theorem 3.1]). Starting here, Weyman constructs in a preprint a universal ring $A(n)_\infty$ and a universal complex modeling the truncation of the last term in the free resolution of an n-generated codimension 4 Gorenstein ideal. The ring $A(n)_\infty$ is naturally a representation of the positive part $\mathbb{L}$ of a Kac–Moody Lie algebra, which has finite type when $n \leq 8$. The representation structure not only encodes the Buchsbaum–Eisenbud multipliers, but also a sequence of *higher structure maps* associated to the free resolutions.

The main contributions of Chapter 5 in the thesis under review refer to the concrete description of the higher structure maps in the case $n = 6$ (which is done in Section 5.3), along with Theorem 5.4.52 (in Section 5.4) which shows that a codimension four Gorenstein ideal with 6 generators is obtained as a section of a codimension 3 ideal with 5 generators (given by submaximal Pfaffians of a 5×5 skew-symmetric matrix by the Buchsbaum–Eisenbud theorem). Under additional hypotheses, this result was previously verified in work of Herzog–Miller [HM85] and Vasconcelos–Villarreal [VV86], but their techniques leave the general case out of reach. Notice that the statement of the theorem does not require any representation theory, and certainly there is no indication that any relationship to Kac–Moody algebras should be relevant for the proof – it is for this reason that I find it to be a remarkable use of representation theory techniques in order to prove a fundamental statement in commutative algebra.

5 Conclusions

The thesis under review addresses fundamental questions at the confluence of representation theory, commutative algebra and algebraic geometry. The thesis is well-written and well-organized, and it provides results that are of interest to a large audience – overall, I find the work to be of *exceptionally high quality*! The author shows a remarkable ability to combine a variety of different techniques, as for instance it is not typical for a researcher in commutative algebra to employ representation-theoretic methods for (possibly infinite dimensional) Lie algebras. In particular Theorem 5.4.52 provides an *original solution to a scientific problem* of current interest, and at the same time an *excellent advertisement* for the use of representation theory techniques in the commutative algebra community. Based on my own understanding and scientific expertise, the thesis in its current form is in compliance with Article 187, Section 2 of the Law of Republic of Poland on Higher Education and Science of July 20, 2018.

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