Review of
"Stability theory for matrix polynomials in one and several variables with extensions of classical theorems"
by Oskar Jakub Szymański
submitted for the PhD degree at Uniwersytet Jagielloński

The doctoral dissertation "Stability theory for matrix polynomials in one and several variables with extensions of classical theorems" by Oskar Jakub Szymański consists of seven chapters, along with a preface, an index of symbols, and a bibliography. The subject of the dissertation is the study of matrix polynomials in both one and in several variables, with a view towards generalizing several classical theorems concerned with bounds on certain classes of polynomials and their zeros. In particular, the work examines several possible generalizations of the notion of stability for a scalar polynomial, with a particular focus on the concept of hyperstability. Many of the theorems and other results contained in the dissertation have previously appeared in the form of a joint journal article, and in a joint preprint.

The main text of the thesis follows a Preface summarizing the contents of the work. Chapter 1, titled Preliminaries, provides some background on aspects of scalar and matrix valued polynomials that is relevant for the thesis. In particular, the notion of D-stability of a polynomial is discussed: a polynomial (in one or in several variables) is stable if $p(z) \neq 0$ in D. Classical one-variable theorems like the Gauss-Lucas theorem and the Szász inequality are stated, and stability-preserving and polarization operations are reviewed. Then, the emphasis is shifted to matrix polynomials. Regularity and singularity notions are presented, and numerical ranges are examined along with various structure results for matrix polynomials and matrix pencils that are used in the thesis. Finally, some recent theorems, which are due to Mehl, Mehrmann, and Wojtylak and have served as an inspiration for the dissertation along with recent work by Knese, are listed.

Chapter 2 introduces the notion of hyperstability for a matrix polynomial; the author of the dissertation singles this out as an important contribution of his work. A polynomial $P(\lambda) \in \mathbf{C}^{n,n}[\lambda]$ is said to be hyperstable in D if for all $x \in \mathbf{C}^n \setminus \{0\}$ there exists $y \in \mathbf{C}^{n,n}$ such that

$$y^*P(\lambda)x \neq 0 \quad \text{for all} \quad \lambda \in D.$$

Hyperstability turns out to imply the more commonly studied notion of stability for matrix polynomials, meaning that if P is hyperstable in D then $\det P(\lambda) \neq 0$ in D. (This definition of stability can be rephrased in similar terms to the displayed condition, but with different quantifiers.) Examples illustrate that a polynomial P can be stable without being hyperstable, but for some classes these notions are shown to coincide. Szymanski discusses how hyperstability behaves under different notions of equivalence; a sample result is that that the orbit $E(\lambda)P(\lambda)F(\lambda)$, where $E, F \in U_n$, never consists exclusively hyperstable polynomials. The last section of Chapter 2 extends the definition of hyperstability to multivariate polynomials.

The next chapter, Chapter 3, deals with a matrix polynomial extension of the classical Gauss-Lucas theorem. Expressed in terms of stable polynomials, the theorem asserts that if $p \in \mathbf{C}[z]$ is stable with respect to a set $D \subset \mathbf{C}$ whose complement is convex, then the derivative p' is stable with respect to D. The direct extension of this result to matrix polynomials fails; however, assuming hyperstability together with a linear independence condition on the entries of $P'(\lambda)$ does result in a generalization of the theorem of Gauss-Lucas in the form of Theorem 3.3. It is shown that both the hyperstability and the linear independence conditions are needed.

Chapter 4 deals with inequalities of Szász type. Theorem 4.1 states that if the matrix polynomial $P(\lambda) = \lambda^d A_d + \cdots + \lambda A_1 + I_n$ has numerical range contained in a half-plane, then its matrix norm satisfies

$$\|P(\lambda)\| \leq 2 \exp\left(\lambda_H[\lambda A_1 - |\lambda|^2 A_2] + \frac{|\lambda|^2}{2}\|A_1\|^2\right);$$

here $\lambda_H(X)$ denotes the largest eigenvalue of $\frac{1}{2}(X + X^*)$. Given this result, it is natural to ask whether hyperstability might be enough to get something similar, but it is shown that this is not the case. This chapter also contains other variants of Szász inequality, assuming for instance certain factorization properties or use of the Frobenius norm. These different inequalities are compared and contrasted, for instance in the very nice Example 4.15.

Chapter 5 contains important sufficient conditions for hyperstability; Szymanski deems these to be central results of his thesis. Theorem 5.1 proceeds, loosely speaking, along the following lines: a quadratic matrix polynomial $P(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ can be certified as hyperstable in D provided one of a list of two-variable matrix polynomials, eg.

$$z_1^2 A_2 + z_2 A_1 + A_0$$

can be shown to be stable with respect to D^2 (possibly along with asking that $0 \notin D$ as for the matrix polynomial $z_1 z_2 A_2 + z_2 A_1 + A_0$). A similar result for cubic polynomials is given by Theorem 5.2, involving stability of three-variable polynomials that are, as is to be expected, more complicated. The author himself mentions that the somewhat lengthy proofs of these theorems contains some repetitions in the different cases treated; perhaps some trimming might have helped the flow of the text.

The next topic, treated in Chapter 6, is that of operators preserving hyperstability. For multivariate matrix polynomials hyperstable in some Cartesian product of half-planes, Proposition 6.1 lists some basic operations that preserve hyperstability. Fixing one of the variables or taking limits need not preserve hyperstability, however, as is demonstrated by example. Polarization does interact well with hyperstability for product domains, as is shown in Theorem 6.4. The short section 6.3 points out that it does not appear to be obvious how to define "singular multivariate matrix polynomials". It would have been interesting to have the author's thoughts about what could potentially be done in this direction laid out here, but perhaps this is the topic of upcoming work.

The final Chapter 7 illustrates how the theory developed in the course of the thesis can be applied to investigate hyperstability for some specific classes of matrix polynomials. For some of the polynomials treated here, stability is easy to establish, but hyperstability does require more work, using results from the dissertation.

The bibliography, while short, contains relevant and appropriate items from the literature. It can be noted that several of the works cited here are quite recent, further illustrating the timeliness of the topic of the dissertation.

The dissertation reviewed here contains both significant new theorems, extensions of previous results, including classical work, and interesting and useful examples and observations. There is also the concept of hyperstability which may well become quite important in future work. I particularly enjoyed the wealth of examples and counterexamples Szymański gave to illustrate how the matrix/several variable setting throws up obstacles that prevent the classical one-variable theory from carrying over immediately, and to illustrate the scope and limitations of the results he obtains. While I'm not a specialist in matrix polynomials, I found that I could follow the flow of the dissertation quite well thanks to its structure and clear exposition. The technical underpinnings of the work do not seem to be excessively complicated, and the proofs are for the most part (see the comment about Chapter 5 above) written out with the appropriate level of detail.

In terms of criticism, I did unfortunately notice some misprints in the thesis; these include in principle easily spotted occurrences of "zeors" (p.19) or "poylnomials" (p.61), etc. Many of these errors could probably have been eliminated in the course of repeated proofreading but perhaps time was a factor here. Similarly, some items in the bibliography seem not to have been updated, viz. Tao's Sendov paper, which has been published in Acta Mathematica some time ago.

Overall, I enjoyed reading this thesis, and I liked the material presented. The problems seem quite natural and the results obtained in this work contribute in a significant way to the theory of stable polynomials in an original way, and suggest several new avenues for future research. In my opinion, Szymański has written a strong doctoral dissertation.

Yours sincerely,
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