GRAPH COLORING GAMES

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A *majority coloring* of a graph is a coloring of its vertices such that for each vertex v, at least half of the neighbors of v have different color than v. Let $\mu(G)$ denote the least number of colors needed for a majority coloring of a graph G. It is well known and easy to prove that $\mu(G) \leq 2$ for every graph G. We consider a game-theoretic variant of this parameter, the *majority game chromatic number* $\mu_g(G)$, defined via a two person coloring game in which one of the players tries to produce a majority coloring of the whole graph, while the other tends to prevent it. We prove that for every integer n there exists a bipartite graph $G(n)$ such that its majority game chromatic number is greater than n. We show construction of such family of graphs. From this it follows that in the general case the parameter $\mu_g(G)$ is unbounded despite its severe limitations in non game version. On the other hand, we indicate that the examined parameter $\mu_g(G)$ is always bounded by $\mathrm{col}_g(G)$, the game coloring number of G. In the next results we improve this bound for some classes of graphs. In particular, we prove that majority game chromatc number is bounded by 3 for every complete binary tree T and also provide example of the tree for which all 3 colors are required. In another theorem we prove that majority game chromatic number of subdivision of any simple graph is as well bounded by 3. The results may suggest that $\mu_g(G)$ is bounded for graphs with bounded coloring number $\mathrm{col}(G)$. However, we also prove that, contrary to this intuition, for every integer k there exists a graph with coloring number equal to 3 and majority game chromatic number greater than k. We provide a construction of such family of graphs.

We also propose a very new variant of coloring game with additional rule that in each moment of the game coloured part of the graph must be a connected subgraph. We define analogical parameter *connected game chromatic number* denoted by $\chi_c(G)$. We show several existing strategies used for the original game that can be adapted to the new modification. We bound this parameter for some special classes of graphs like the family of bipartite graphs, showing that new variant is a significantly differrent game from its original counterpart. We also prove that for every outerplanar graph its connected game chromatic number is always less than or equal to 6. We improve this bound for triangulated outerplanar graphs showing that for such class, connected game chromatic number is bounded by 5. We also present example of triangulated outerplanar graph with connected game chromatic number equal to 5, showing that the last bound is tight. Our last result shows that for every integer k there exists a graph with coloring number equal to 4 and connected game chromatic number greater than k. Again we are giving precise construction of such family of graphs. Based on the examined families of graphs, we pose a conjecture that for any graph, its connected game chromatic number is not greater than its classical counterpart: $\chi_c(G) \leq \chi_g(G)$.

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