Review of doctoral dissertation for the Jagiellonian University of Kraków

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Title of dissertation: Cyclic vectors in Dirichlet-type spaces in the unit ball of $\mathbb{C}^n$

Name of reviewer: Thomas RANSFORD (Université Laval, Québec)

1. Background

Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$. Given $n \geq 2$, we write $\mathbb{D}^n$ for the unit polydisk in $\mathbb{C}^n$ and $\mathbb{B}_n$ for the unit ball in $\mathbb{C}^n$.

For $\alpha \in \mathbb{R}$, the Dirichlet-type space $D_\alpha(\mathbb{B}_n)$ is the Hilbert space of holomorphic functions $f(z) = \sum_{|k|=0}^{\infty} a_k z^k$ on $\mathbb{B}_n$ such that

$$\|f\|^2_{D_\alpha(\mathbb{B}_n)} := \sum_{|k|=0}^{\infty} (n+|k|)^\alpha \frac{(n-1)!k!}{(n-1+|k|)!}|a_k|^2 < \infty.$$

Here we have used the usual multi-index notation to represent $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$.

Although $D_\alpha(\mathbb{B}_n)$ is not an algebra if $\alpha \leq n$, it is still closed under multiplication by polynomials, so it makes sense to consider cyclic functions, namely functions whose polynomial multiples are dense in $D_\alpha(\mathbb{B}_n)$. Cyclic functions play a similar role in these spaces to that played by invertible elements in algebras. It is therefore of great interest to determine exactly which functions are cyclic.

This turns out to be a difficult problem, even in the case $n = 1$. In that case, the cyclic functions are known when $\alpha \leq 0$ or $\alpha > 1$, but not if $\alpha \in (0, 1]$. In particular, when $\alpha = 1$ (the classical Dirichlet space $D_1(\mathbb{D})$), the characterization of cyclic functions is the subject of a famous conjecture of Brown and Shields [11], which has remained an open problem for nearly 40 years.

It is thus natural to focus attention on the cyclicity of polynomials, which are somewhat easier to treat. When $n = 1$, the characterization of cyclicity of polynomials in $D_\alpha(\mathbb{D})$ is easy and well understood. Cyclicity of polynomials in $D_\alpha(\mathbb{D}^2)$ was characterized by Bénéteau *et al* in [7], and some partial results in $D_\alpha(\mathbb{D}^n)$ for $n \geq 3$ were obtained by Bergqvist in [8].

The subject of the dissertation is the problem of characterizing cyclicity of polynomials in $D_\alpha(\mathbb{B}_n)$ when $n \geq 2$. Some partial results were previously obtained by Sola in [35].

2. Content of the dissertation

The main result of the dissertation is a complete characterization of the cyclicity of polynomials in $D_\alpha(\mathbb{B}_2)$. Since a product of polynomials $p_1 \dots p_k$ is cyclic iff each p_j is cyclic, and all cyclic polynomials must be zero-free in $\mathbb{B}_2$, it is sufficient to treat the case of irreducible polynomials that are zero-free in $\mathbb{B}_2$.

The following result is stated as an unnumbered result in the introduction. It is obtained by combining Theorems 5.0.3 and 7.0.1 from the main text.

Theorem A. *Let $p \in \mathbb{C}[z,w]$ be an irreducible polynomial that is zero-free in $\mathbb{B}_2$.*

(i) *If $\alpha \leq 3/2$, then p is cyclic for $D_\alpha(\mathbb{B}_2)$.*

(ii) *If $3/2 < \alpha \leq 2$, then p is cyclic in $D_\alpha(\mathbb{B}_2)$ iff it has finitely many zeros in $\partial\mathbb{B}_2$.*

(iii) *If $\alpha > 2$, then p is cyclic in $D_\alpha(\mathbb{B}_2)$ iff it is zero-free in $\partial\mathbb{B}_2$.*

This theorem is the subject of the paper [23], co-authored by the candidate and his supervisor, and now accepted in *Constructive Approximation.*

The question of cyclicity in $D_\alpha(\mathbb{B}_n)$ for $n \geq 3$ remains open. There are however some partial results. The second main result of the dissertation concerns so-called model polynomials, namely polynomials of the form

$$\pi_m(z_1, \dots, z_n) := 1 - m^{m/2} z_1 z_2 \cdots z_m \quad (1 \leq m \leq n).$$

These had previously been studied when $m = 1, 2$. The following theorem is obtained by combining Theorem 3.2.1 and Lemma 4.2.2.

Theorem B. *The model polynomial π_m is cyclic in $D_\alpha(\mathbb{B}_n)$ iff $\alpha \leq (2n + 1 - m)/2$.*

This theorem is the subject of the paper [37], authored sole by the candidate, and now accepted in the *Canadian Mathematical Bulletin.*

The dissertation concludes with some ideas for characterizing cyclicity of general polynomials in $D_\alpha(\mathbb{B}_n)$ for $n \geq 3$. Theorems A and B, and the techniques used to establish them, lead the author to formulate the following problem (see Remark 11.0.2).

Open problem. *Let $p \in \mathbb{C}[z_1, \dots, z_n]$ be a polynomial that is zero-free in $\mathbb{B}_n$. Suppose that the intersection of the zero set of p with $\partial\mathbb{B}_n$ contains a real submanifold of $\mathbb{R}^{2n}$ of dimension $m - 1$, but no submanifold of any higher dimension. Then p is cyclic in $D_\alpha(\mathbb{B}_n)$ iff $\alpha \leq (2n + 1 - m)/2$.*

3. Evaluation of the dissertation

(a) Content. The two main theorems, listed as Theorems A and B above, are deep and interesting results. Proving them is hard work, and requires the exploitation of a number of advanced techniques. Some of these, notably diagonal subspaces, Cauchy transforms and radial dilations, are familiar to me, though in this thesis they appear to have been pushed further than ever before. Others, in particular the ideas from semi-analytic geometry in Section 2.2 and their application in conjunction with capacity criteria, as in Section 7, are completely new to me. They appear to be the added ingredient needed to push the previously known partial results all the way to the definitive result stated in Theorem A above. The open problem stated at the end points to interesting avenues of future research. This is a very impressive piece of work.

(b) Presentation. The thesis is well presented for the most part. Detailed explanations are given where necessary. The language is somewhat lacking in style, and there are occasional grammatical mistakes, probably due to the fact that English is not the candidate's mother tongue. A detailed list of corrections is given in an appendix to this review. The typesetting is adequate. The bibliography is complete and up to date, and presented according to the usual norms in the subject. I found the thesis relatively easy to follow, and actually quite enjoyed reading it.

4. Conclusions

In my view, the thesis amply fulfils the requirements for a doctorate.

I would go further, and say that the excellence of the results obtained certainly merits a distinction. My only reservation is that a part of the work was carried out by the candidate in collaboration with his supervisor (at least, as evidenced by their joint publication). What effect this should have, if any, I leave to the judgement of the Board of Discipline of Mathematics.

T J Ransford ———— 11 July 2023

Thomas Ransford ———— Date

Appendix: Detailed comments

page	*line*	
7	15	'e.t.c.' should be 'etc.'
8	6	'converse'
8	7	Give a reference for the Brown–Shields conjecture.
8	9	'several-variable'
8	10	Remove the 'on'
8	13	'a' should be 'α'
8	19	Replace 'solved' by 'obtained'
8	-12	Replace 'anisotropic one' by 'anisotropic case'.
9	3	'a' should be 'α'
13	-1	'$n \leq 3$' should be '$n \geq 3$'
14	5	'natural numbers'
14	-4	'n-tuple' (the dollar signs were omitted)
15	7	Replace 'will denote either' by 'signifies'.
15	8	Delete 'tends to a positive constant as k tends to infinity or it'.
15	19	In equation (1.1.3), the integral on the left-hand side should be over $\mathbb{B}_n$, not $\mathbb{C}^n$, and the upper limit in the integral on the right-hand side should be 1, not ∞.
15	-2	Give a more a precise reference ([32] is a book).
18	2	'well-known'
18	9	Insert a comma after '$H^\infty(\mathbb{B}_n)$'.
18	12	'introductions'
18	-5	'Möbius-invariant'
18	-5	Give a reference for the fact that the Dirichlet space is the unique Möbius-invariant Hilbert space in $\mathbb{B}_n$.
18	-2	Define the Drury–Arveson spaces, and explain their relevance here.
19	-8	Maybe it might be helpful to recall the definition of the operator R.
19	-5	It would make more logical sense to say 'where $q \in \mathbb{N}, v = \alpha - 2q, \ldots$'
19	-1	There seems to be a mistake in this formula. I think that the left-hand side should be $$\sum_{j=0}^{n} \binom{q}{j} n^j R^{n-j}(f).$$

20	4	This formula also needs to be corrected.
20	8	The statement of Theorem 1.3.2 is hard to read. I suggest: 'If $\alpha < 1$, if $\delta, \tau > -1$, and if $\min(\delta, \tau) + \alpha > -1$, then...'
20	16	For the proof of Lemma 1.3.3, it is enough to give the reference. The rest is unnecessary repetition.
22	-11	There is a sum '$\sum_p$' missing at the beginning of the term on the right-hand side.
23	1	It seems obvious that $\mathcal{N}_\alpha$ satisfies the triangle inequality, since it is given by an inner product. Am I missing something?
24	4	The sentence following 'In particular' is true, but it is not a consequence of what precedes it, so it is misleading to say 'In particular'.
24	-17	Not only must a multiplier of $D_\alpha(\mathbb{B}_n)$ be holomorphic in $\mathbb{B}_n$, but also it must belong to $D_\alpha(\mathbb{B}_n)$.
26	9	'Theorem 1.3.1'? There is no Theorem 1.3.1.
26	13	Replace 'are coming up' by 'often arise'.
26	19	'of f' (the dollar signs were omitted)
26	-14	Give a reference for the fact that, if f is holomorphic in $\mathbb{B}_n$ and $\mathcal{K}f$ is zero on a set of positive measure, then $f \equiv 0$. (This is non-trivial even if $n = 1$.)
26	-7	Maybe this would be a good place to mention the Drury–Arveson spaces.
27	6	Replace 'reflexible' by 'reflexive'. The same error needs to be corrected throughout the text.
27	16	Replace 'pointwisely' by 'pointwise'. The same error needs to be corrected throughout the text.
27	18	'Recall ...'. This fact was already used above (see line 10).
27	23	The Lemma should be numbered.
27	25	The proof of the lemma is very laborious. There is no need to talk about weak*-convergence (as opposed to weak convergence), since $D_\alpha(\mathbb{B}_n)$, being a Hilbert space, is reflexive anyway. The only non-trivial point in the proof is the recourse to Proposition 2 of [11]. It would be better to spend more time explaining this latter result.
28	19	'Recall that a convex subset of a normed space is closed if and only it is sequentially weakly closed'. Give a reference for this fact.
28	-1	Explain the relevance of Drury–Arveson spaces here.
30	-14	Replace '$\dim E$' by '$\dim_{\mathbb{R}} E$'.

31	19	Replace ‘vanish’ by ‘vanishes’.
34	-11	Reword as follows: ‘Given f, the holomorphic function $\tilde{f}$ is unique’.
34	-2	This is very confusing, until one realizes that $f'(z)$ does *not* mean the derivative of f. Use another notation instead of f'.
35	4	Replace ‘settings’ by ‘setting’.
36	1	Replace ‘settings’ by ‘setting’.
36	5	Replace ‘admit’ by ‘admits’.
36	5	Replace ‘to a strictly bigger domain’ by ‘to a ball of strictly bigger radius’. (As it stands, the phrase is not sufficiently precise.)
36	-9	There is something missing in this formula. Maybe ‘$\theta_i \in [0, 2\pi]$’?
37	6	In the definition of the Cauchy transform, should it be ‘$\overline{\zeta}$’ or ‘ζ’?
38	3	Make it clear that ‘supported on’ means ‘supported on a compact subset of’.
38	9	Replace ‘exist’ by ‘exists’.
38	11	‘The theory regarding ...’
38	-3	Replace ‘setting’ by ‘considering’.
39	-4	‘It is easy to see that...’. This explanation is too brief: it should be expanded.
40	4	Replace ‘overcome’ by ‘treated’.
40	7	Replace ‘non-zero’ by ‘nowhere-vanishing’.
40	11	‘$\mathrm{dist}(z, \mathcal{Z}(f))$’ should be ‘$\mathrm{dist}(x, \mathcal{Z}(f))$’
41	0	The chapter heading is too long to fit at the top of the page.
41	-11	Replace ‘Set’ by ‘Consider’.
42	8	The phrase ‘$\|p/p_r\|_{D_\alpha(\mathbb{B}_2)} < \infty$ as $r \to 1^-$’ is very sloppy. This quantity is obviously finite. What is really meant here is that it remains bounded as $r \to 1^-$. Maybe write instead $\limsup_{r \to 1^-} \|p/p_r\|_{D_\alpha(\mathbb{B}_2)} < \infty.$ The same mistake is made several times, and needs to be corrected throughout the text.
42	-16	In fact, for $\alpha < 3/2$ we even get norm convergence, since weak convergence transforms to norm convergence under a compact linear map.
42	-14	Replace ‘show’ by ‘showing’.

42	-11	Replace 'is one' by 'is each'.
42	-2	Replace 'An idea' by 'The idea'.
43	-5	Replace 'in hands' by 'in hand'.
44	2	The alignment of the displayed formulas is a mess. The equals signs should be aligned vertically.
44	-12	Replace 'Functions' by 'The functions'.
45	1	'the above lemma says that...' Explain why. It is not obvious.
47	3	Missing full stop at the end of the formula.
47	11	Replace 'estimates' by 'estimate'.
48	8	There is an unnecessary break in the formula.
48	-1	Align the plus signs.
49	12,14	Once again, these should be written $\limsup_{r\to 1^-} \int(\cdots) < \infty$.
49	-2	There is an unnecessary break in the formula.
50	11,15	Replace 'is bounded' by 'remains bounded' (twice).
51	-2	Replace 'is bounded' by 'remains bounded'.
53	9	Once again, this should be written $\limsup_{r\to 1^-} \|(\cdots)\| < \infty$.
53	-9	Replace 'are finite' by 'remain bounded'.
55	-3	Replace 'the one above estimates' by 'the estimates above'.
56		General comment about Section 9: This section is not as well written as the others. There is a significant amount of repetition at the beginning. Also, the results obtained were already found by other means earlier in the thesis. The purpose of this section should be clarified.
63	-1	This section ends rather abruptly. What is the conclusion?
64	10,12	Replace 'vanish' by 'vanishes' (twice).
70	[34]	The volume number is missing from the article of Sargent and Sola.