Review of
"Cyclic vectors in Dirichlet-type spaces in the unit ball of $\mathbf{C}^n$"
by Dimitrios Vavitsas
submitted for the PhD degree at Uniwersytet Jagielloński

The doctoral dissertation "Cyclic vectors in Dirichlet-type spaces in the unit ball of $\mathbf{C}^n$" by Dimitrios Vavitsas consists of eleven chapters, along with an introduction and abstract, an index of symbols, and a bibliography. The subject of the dissertation is the study of vectors in certain function spaces in the unit ball in $\mathbf{C}^n$ that are cyclic with respect to the multiplication operators induced by coordinate functions, with a special emphasis on n-variable polynomials that are zero-free in the unit ball.

The main text of the thesis follows a short Introduction and Abstract summarizing the contents of the work. Chapter 1 provides some background and notation on holomorphic function spaces on the unit ball. In Subsection 1.1, representations of integrals over the ball are recalled, along with facts about automorphisms, and the anisotropic distance that is adapted to function theory in the ball. Then, in Subsection 1.2, the Dirichlet-type spaces are discussed. Initially D_α is introduced in terms of a power series norm: we say a holomorphic function $f\colon \mathbf{B}^n \to \mathbf{C}$ having power series expansion $f = \sum_k a_k z^k$ belongs to $D_\alpha(\mathbf{B}^n)$ if the series

$$\|f\|^2_{D_\alpha} = \sum_k (n+|k|)^\alpha \frac{(n-1)!k!}{(n-1+|k|)}|a_k|^2$$

is convergent. Here, standard multi-index notation is used. A brief but interesting discussion about possible extensions to more general Reinhardt domains follows. Here, there are some ideas that could lead to further work. Then, in Subsection 1.3, equivalent norms for D_α are discussed. In particular, integral norms involving gradients and radial derivatives are used throughout the thesis as a technical tool for obtaining bounds and estimates that seem harder to find using the series norm. Further structural and functional analytic properties of Dirichlet-type spaces are listed in Subsections 1.4 and 1.5. In particular, there is an interesting discussion in Subsection 1.4 concerning multiplier operators on D_α: some of these results may be new, and in any case it is good to see these statements gathered in one place. The main problem addressed in the thesis is set out in Subsection 1.5. Namely, a fixed vector $f \in D_\alpha(\mathbf{B}^n)$ is said to be cyclic if the closed subspace

$$[f]_\alpha = \mathrm{clos}_{D_\alpha}\mathrm{span}\{z_1^{k_1}\cdots z_n^{k_n} f \colon n_j \in \mathbf{N}\}$$

is equal to all of $\mathrm{D}_\alpha(\mathbf{B}^n)$; in other words, the smallest $z_1, \ldots, z_n$-invariant subspace of D_α that contains the vector f is the whole space. Since point evaluations $e_{z_0}\colon \mathrm{D}_\alpha \to \mathbf{C}$ furnish continuous linear functionals on each D_α, no cyclic function can have $Z(f) \cap \mathbf{B}^n \neq \emptyset$. A significant difficulty arises, however, when f has no zeros in an n-ball, but some zeros in the unit sphere $\mathbf{S}^n$: then it is not immediately clear whether or not f is cyclic in some range of $D_\alpha(\mathbf{B}^n)$.

Naturally following the formulation of the cyclicity problem, Chapter 2 deals with zero sets of D_α-functions, with a special emphasis on zeros inside the unit sphere $\mathbf{S}^n$, the boundary of the unit ball. Some geometric notions, like complex-tangential curves, are reviewed in Subsection

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2.1, and some results from interpolation theory are listed. Boundary zeros of polynomials are examined in detail in Subsection 2.2, and some important results on their structural properties are proved. Of particular importance is the observation that the boundary zero set $\mathbb{S}^n \cap Z(p)$ of a polynomial is either finite or, in a certain sense, contains a piece of a graph of an analytic function. Given the importance of these results for the rest of the dissertation, and the selection of functional analysis material that is reviewed in Chapter 1, one might have hoped for a little more background discussion of these ideas from geometry that may be less well known to analysts.

Chapter 3 contains contributions to cyclicity in $D_\alpha(\mathbb{B}^n)$ for $n \geq 2$. It is observed that for functions of the form $f(z_1, \ldots, z_n) = F(z_1 \cdots z_n)$, where F is a one-variable function, questions about cyclicity in the n-dimensional ball can be reduced to cyclicity in $D_{A(\alpha)}(\mathbb{B}^1)$, a certain one-variable Dirichlet-type space. In particular, we encounter the model polynomials $1 - m^{m/2} z_1 \cdots z_m$, $m \leq n$, that play an important role in later chapters; here it is shown that they are cyclic whenever $\alpha \leq \frac{2n+1-m}{2}$. The results in Chapter 3 mirror those in earlier work of Bénéteau et al in the setting of Dirichlet spaces in the bidisk, and in a previous paper of the reviewer dealing with the 2-ball. While the translation of these statements to the n-ball is straight-forward, the proofs do require some non-trivial adaptations from those appearing in previous work; for instance, one needs to pay attention to how many variables m out of n possible choices feature in a model polynomial.

Chapter 4 introduces some notions from potential theory and applies them to the study of cyclic vectors. As was noted when $n = 2$ in the reviewer's paper, it is observed for $n \geq 2$ that if the boundary zero set of a Dirichlet function f has positive α-capacity, then f is not cyclic in D_α. The notion of capacity used here is a type of anisotropic α-capacity associated with kernels of the form $|1 - \langle z, w\rangle|^{\alpha-n}$ (with appropriate logarithmic expression when $\alpha = n$). These considerations are enough to show that $1 - m^{m/2} z_1 \cdots z_n$ is not cyclic when $\alpha > \frac{2n+1-m}{2}$. The methods of proof in Chapter 4 are again quite similar to previous work, and the exposition is a little on the sketchy side in certain places. For instance one could have asked for a more detailed description of the measure whose Cauchy transform is used to show non-cyclicity.

Chapter 5 contains the interesting result that any polynomial with at most finitely many zeros in $\mathbb{S}^n$ and no zeros in the ball is cyclic in D_α whenever $\alpha \leq n$. The idea in the proof is to use Łojasiewicz's inequality to reduce to the case of products of the model polynomial $1 - z_1$ and its images under composition by unitaries. The same scheme was used by Bénéteau et al and by Knese et al in the polydisk setting, but here the geometry of the ball enters, forcing some additional steps to be taken compared to the case of $\mathbb{D}^n$.

Chapter 6 can be viewed as one of the highlights of the dissertation. Here, it is proved that any polynomial p in two variables with no zeros in $\mathbb{B}^2$ is cyclic in D_α for $\alpha \leq 3/2$. The approach taken is to look at p/p_r, where $p_r(z_1, z_2) = p(rz_1, rz_2)$, and to prove that the norms $\|p/p_r\|_\alpha$ remain bounded as $r \to 1$. This is, by general functional analysis and the fact that $1/p_r$ are multipliers acting on D_α, enough to show that 1 belongs to the closed invariant subspace generated by p, showing turn that p is cyclic. To achieve the desired norm boundedness, Puiseux factorization of the polynomial p is employed, and various fairly complicated expressions are estimated in the integral versions of the Dirichlet norm. All in all, the proof is quite demanding (eg. one needs to keep track of many different terms of similar but not identical type), and in the reviewer's opinion, impressive.

The short Chapter 7 establishes a reverse implication, namely that if the zero set of the polynomial p in $\mathbb{S}^2$ contains infinitely many points, then p is not cyclic in D_α whenever $\alpha > 3/2$. The trick is to appeal to results from Chapter 2 and to show that if the zero set $Z(p) \cap \mathbb{S}^2$ is infinite, then it actually contains a curve that can be transformed into the zero set $Z(1-2z_1z_2)$. The model polynomial $1 - 2z_1z_2$ was seen to fail to be cyclic when $\alpha > 3/2$ in Chapter 4, and the non-cyclicity of p follows. This is a short but clever argument.

Chapter 8 returns to the n-dimensional setting. Using the radial dilation method as in Chapter 6, it is shown that the model polynomials $p = 1 - m^{m/2}z_1 \cdots z_m$ are cyclic for $\alpha \leq \frac{2n+1-m}{2}$. The simple form of p here allows for explicit computations.

By contrast, Chapter 9 deals with Dirichlet-type spaces in the unit disk. The first parts of this chapter are standard and could well have been moved to the first half of the thesis. However, the second half of Chapter 9 contains a lemma that gives an interesting formula for the $D_\alpha(\mathbb{B}^1)$-norm of f/f_r when f is now a general outer function and f_r is its dilation. Since outer functions are cyclic in D_0 in this setting, such choices of f are natural candidates for cyclicity. While the formula obtained in the lemma is not simple enough to lead to any obvious breakthroughs, it could potentially point the way to a different way of approaching the cyclicity problem in one variable for general functions. This reviewer hopes that this direction is pursued in future work.

Chapter 10 attemps to obtain bounds on n-variable radial dilations in the integral norm for Dirichlet-type spaces in the ball. A lemma is presented, which again gives a double-sum formula for the relevant integrals in terms of quantities related to the target polynomial p. Again, the resulting formula is sufficiently complicated that it is not readily apparent to the reviewer how it can be employed in practice to investigate cyclicity of some given polynomials, but that does of course not rule out the possibility that the approach taken here could bear fruit in future work.

The final Chapter 11 lists some interesting open problems and remarks about similarities and differences between the ball and the polydisk versions of the cyclicity problem for Dirichlet spaces.

The bibliography contains relevant and appropriate items from the literature and gives the reader a good entry point for background and further reading.

The dissertation by Vavitsas reviewed here addresses very natural and important problems in function theory and operator theory. In the case of the 2-ball, cyclic polynomials are completely characterized as a result of the work in the thesis, an impressive achievement. This reviewer has worked on precisely this question, and had estimated this to be a difficult problem. Certainly, considerable ingenuity was needed to resolve the problem in the thesis. The n-dimensional situation is also explored by Vavitsas, and while the results there are less complete, they will no doubt be useful in further study. (There is a suspicion, supported by some of the results in this thesis, that any characterization in higher dimensions will be more involved than in two variables.) Throughout the work, Vavitsas demonstrates a good command of research-level complex analysis, functional analysis, and complex geometry.

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Apart from some minor language issues and, and as remarked before, some perhaps unbalanced choices in the choice of what background material is provided and discussed, the exposition is certainly satisfactory.

In summary, this is, in this reviewer's opinion, a strong doctoral dissertation worthy of a distinction.

Yours sincerely,
Alan Sola
Department of Mathematics
Stockholm University

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