DISSERTATION SUMMARY
MEDIAL AXIS AND SINGULARITIES

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The subject of the doctoral dissertation is the study of the medial axis of a closed set on the grounds of singularity theory. The medial axis of the closed set X, denoted by M_X, consists of those ambient space points for which more than one closest element in X exists. In symbols that is $M_X := \{a \in \mathbb{R}^n |\ \#m(a) > 1\}$, where $m(a)$ denotes the set of points in X closest to a.

The beginnings of the notion reach back to the mid-twentieth century. In the influential work [Fed59], Herbert Federer studied sets that exhibit properties shared by convex sets and differentiable manifolds. Without defining the medial axis per se, he distinguished actually a class of sets separated from their medial axis closure. For those quasi-convex sets, he constructed geometric measures and proved a series of results involving the remarkable Steiner formula describing the measure of the Minkowski sum $X + \mathbb{B}(0, r)$.

However, the title of the medial axes father belongs to Henry Blum. In [Blu67], he noted that the set so scrupulously avoided by Federer is a complete shape descriptor. This observation opened the medial axes for the applications in the image recognition field, and therefore also in robotics, computer tomography and others. Surprisingly, however, the mathematical theory of the medial axis still awaits a thorough description. The theory's nomenclature is especially discrepant – the medial axes appear under the names of skeletons, cut loci or even wrongly central sets. Furthermore, the applicative motivation routinely restricts the results to the planar subsets or the sets of codimension one.

Even though it seems natural considering the Federer lineage, the link between the medial axes and singularity theory became the subject of research only recently in the paper by Lev Birbrair and Maciej Denkowski [BD17b; BD17a]. In this work, the authors presented a compelling vision of detecting a set's singularities by examining its medial axis. In the context restricted to planar subsets, they characterised the intersection $X \cap \overline{M_X}$. Also, they introduced a series of tools serving the measurement of the distance between the medial axis and the set points.

The dissertation concentrates on adapting the medial axis theory to arbitrary dimensional subsets of Euclidean spaces definable in o-minimal structures. One of the first significant results is the description of the medial axis tangent cone by the multifunction of the closest points. *For $a \in M_X$ we have $M_{m(a)} \subset C_a M_X$.* The result corresponds to analogous theorems from the conflict set theory, obtained by Lev Birbrair and Dirk Siersma in [BS09]. Despite the alluring similarities between the medial axis and the conflict set, the methods of Birbrair and Siersma are not transferable to the medial axis theory. We are bound, therefore, to develop a novel approach.

The consequences of the results concerning the tangent cone touch the dimension theory. In the second result section, we describe the dimension of the medial axis in terms of the dimension of the closest points multifunction. It is somewhat equivalent to the Remmert rank theorem, known from complex analytic geometry, enriched by the typical medial axis twist. It appears that the local dimension of the medial axis at a point a depends not only on the points closest to a but also on the dimension of $m(b)$ for b in a neighbourhood of a. *For any closed set $X \subset \mathbb{R}^n$ and a point $a \in M_X$ there is* *an equality* $\dim_a M_X + \min\{\dim m(b) \,|\, b \in U,\ U$ *is a neighbourhood of* $a\} = n - 1$. Naturally, the local results lead to the global formula. *For any closed* $X \in \mathbb{R}^n$ *there is* $\dim M_X + \min\{\dim m(a) \,|\, a \in M_X\} = n - 1$. At the same time, we answer in the o-minimal geometry the conjectures of Erdös [Erd45; Erd46] and Denkowski [Den11].

The next object of the dissertation study is the reaching radius $\tilde{r}(a)$ - an equivalent of the curvature radius for singular sets. The radius defined in the thesis bases on the ideas of Birbrair and Denkowski from [BD17b]; however, it reverses the order of taking the limits. With its help, we characterise the frontier points of the medial axis. *For any* $a \in \mathbb{R}^n$ *there is* $a \in \overline{M_X} \setminus M_X$ *if and only if* $\tilde{r}(a) \leq d(a, X)$ *and* $\# m(a) = 1$. This characterisation is a substantial strengthening of the results of Tatsuya Miura [Miu16]. The result stands out in the sense that it holds outside the o-minimal structures. The subsequent achievement of the reaching radius is the simplification of the original definitions of Birbrair and Denkowski and the proof of the continuity of their radius on the $\mathscr{C}^2$-smooth part of the set. Lastly, by applying the results, we prove that the order of taking limits in the Birbrair-Denkowski definition does not change the outcome value of the reaching radius.

The final topic raised in the thesis is the problem of the intersection of the set and its medial axis closure. In this scope, we can restrict the attention to the $\mathscr{C}^2$-singular part of the set. Indeed, according to the John Nash results [Nas52], the medial axis is separated from the $\mathscr{C}^2$-smooth part of the set. We split the problem into natural subcases of $\mathscr{C}^1$-smooth and $\mathscr{C}^1$-singular parts of the set. In the first case, we obtain the generalised version of the theorem from [BD17b]. Unfortunately, the higher dimensional case proves to be more sophisticated than the planar one. *The superquadracity - a tool used by Birbrair and Denkowski - ceases to be a necessary condition for detecting the approaching medial axis (however, it remains sufficient).* In the scope of the second case, we obtain two series of results. On the one hand, we strengthen the tangent cone criterion of Birbrair and Denkowski. *For any* $a \in X$ *with* $C_a X \not\subset \liminf_{b \to a} C_b X$, *there is* $a \in \overline{M_X}$. On the other, we generalise the results of Jan Rataj and Ludek Zajíček [RZ16]. *For any* $\mathscr{C}^1$*-singular point* $a \in X$ *such that there exists a* $\dim_a X$*-dimensional topological submanifold* $\Gamma \subset X$ *containing* a, *there is* $a \in \overline{M_X}$.

References

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