REPORT ON PHD THESIS OF SZYMON MYGA

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The title of the thesis is "Convex and symplectic properties of Monge-Ampère-type operators on compact Kähler manifolds with holomorphic torus actions". It builds on the papers [Myg21a] and [Myg21b], but as the thesis is written as a monograph, in addition to the original work of the two papers, it also contains a generous introduction to the subject of torus actions on compact Kähler manifolds and the associated g-Monge-Ampère equation.

1. BACKGROUND MATERIAL

I will start by recalling some basic definitions and results that provide the context to the original work. All of this is explained very nicely in sections 1-5 of the thesis.

1.1. **Symplectic manifolds.** Let M be a smooth manifold of even real dimension $2n$. A two-form ω on M is said to be symplectic if it closed and non-degenerate. The pair (M, ω) is then called a symplectic manifold, and we see that it comes with a prescribed volume form ω^n (or sometimes $\omega^n/n!$).

1.2. **Momentum maps.** Assume that a Lie group G acts on M leaving ω invariant. Note that any left-invariant vector field X on G induces a vector field $X^\#$ on M. The action is said to be Hamiltonian if there exists a so called momentum map $\mathbf{m}$ from M to the dual of the Lie algebra of G such that for any X:

$$d\langle \mathbf{m}(x), X\rangle = \omega_x(X^\#, \cdot).$$

The momentum map is typically also assumed to be G-invariant.

1.3. **Momentum polytopes.** A celebrated result of Atiyah [Ati82] and independently Guilleman-Sternberg [GS82] says that when M is compact and G is a compact torus $\mathbf{T}^k$ then the image of the momentum map is a compact convex polytope Δ. If the dimension of the torus is n, making the action completely integrable, then Delzant proved that Δ is "Delzant" [Del88], and in particular M carries the structure of a toric projective variety.

When M is toric, the pushforward of the volume form ω^n by the momentum map gives the Lebesgue measure on Δ.

1.4. **Symplectic reduction.** To understand what happens with the pushforward of the volume form ω^n by the momentum map when the action is not completely integrable one needs to introduce the key notion of symplectic reduction, due to Marsden-Weinstein [MW74].

Let p be a regular value of the momentum map and assume that T^k acts freely on $\mathbf{m}^{-1}(p)$. Then $M_p := \mathbf{m}^{-1}(p)/\mathbf{T}^k$ is a manifold called the reduced space, and interestingly it inherits a symplectic structure σ_p from M, by demanding that $\pi^*\sigma_p = i^*\omega$, where π is the projection from $\mathbf{m}^{-1}(p)$ to M_p and i is the inclusion of $\mathbf{m}^{-1}(p)$ in M.

1.5. **Duistermaat-Heckman measure.** We can now come back to the question of what happens if one pushes forward to volume form ω^n by the momentum map to the moment polytope. Duistermaat-Heckman [DH83] showed that we get a measure on Δ whose density at a point p is equal to $\int_{M_p} \sigma_p^{n-k}$ (up to a dimensional constant). The also described how σ_p varies with p, implying that the function $\int_{M_p} \sigma_p^{n-k}$ is piecewise polynomial in p.

1.6. **Kähler manifolds.** We now assume that (M, ω) is not only symplectic, but even Kähler, meaning that it has a complex structure adapted to the symplectic one. In particular, locally one can then find a potential of ω, i.e. a function u such that $\omega = i\partial\bar{\partial}u$.

1.7. **Kähler potentials and ω-psh functions.** A smooth function ϕ on M is said to be a Kähler potential if $\omega_\phi := \omega + i\partial\bar{\partial}\phi$ is Kähler. The space of Kähler potentials is denoted by $\mathcal{H}(M, \omega)$. A weaker notion is that of a ω-psh function. ϕ is said to be ω-psh if locally $u + \psi$ is psh (where u is a local potential for ω). We then see that $\omega_\phi := \omega + i\partial\bar{\partial}\phi$ is not necessarily a Kähler form but a closed positive $(1,1)$-current. The space of ω-psh functions is denoted by $PSH(M, \omega)$.

1.8. **Monge-Ampère measures.** The (complex) Monge-Ampère measure $MA_\omega(\phi)$ of a C^2 ω-psh function ϕ is defined as ω_ϕ^n. Since the wedge products of currents need not be well-defined, when ϕ is singular special care has to be taken to give meaning to $MA_\omega(\phi)$. For ϕ locally bounded this was achieved by Bedford-Taylor [BT82]. For more general ϕ Boucksom-Eyssidiuex-Guedj-Zeriahi [BEGZ10] introduced the so-called non-pluripolar Monge-Ampère, here also denoted by $MA_\omega(\phi)$. For ϕ locally bounded we have that

$$\int_M MA_\omega(\phi) = \int_M \omega^n,$$

but when ϕ is not locally bounded it can happen that

$$\int_M MA_\omega(\phi) < \int_M \omega^n.$$

ϕ is said to have full mass if there is equality, and the space of full mass potentials is denoted by $\mathcal{E}(M, \omega)$.

1.9. **Optimal transport and the (real) g-Monge-Ampère equation.** Given two measures μ and ν on $\mathbb{R}^n$ the optimal transport problem seeks to find the map T which transports μ to ν, i.e. such that $T_*\mu = \nu$, and which among all those transport maps minimizes the cost

$$\int_{\mathbb{R}^n} c(x, Tx) d\mu(x),$$

where $c(x, y)$ is some given cost function. Here we will focus on the case of the quadratic cost function $c(x, y) := |x - y|^2$. It turns out that, given some mild assumptions on μ and ν, the optimal transport map will be of the form $\nabla\phi$ where ϕ is a convex function. That $\nabla\phi$ transports $\mu = f(x)dx$ to $\nu = g(y)dy$ can be rewritten as the so-called g-Monge-Ampère equation

$$g(\nabla\phi) \det D^2\phi = f.$$

A fundamental theorem of McCann [McC95] says that as long as μ and ν are probability measures and μ vanishes on Borel subsets of Hausdorff dimension less than or equal to $n - 1$, then there is a transport function of the form $\partial\phi$ (i.e. we take the subgradient map, since the gradient might not be defined everywhere), and $\partial\phi$ is uniquely determined μ-everywhere.

2. Torus actions on compact Kähler manifolds

I will now discuss the results presented in Section 6 of the thesis.

2.1. **Principle $\mathbf{T}^k_{\mathbb{C}}$-bundle.** Now assume that there is a Hamiltonian $\mathbf{T}^k$-action on M. As explained in Section 5 of the thesis this action can be extended to a holomorphic action of the complex torus $\mathbf{T}^k_{\mathbb{C}}$. It is also shown in Section 5 that an open dense subset of M, here denoted M^0, will have the structure of a holomorphic principle $\mathbf{T}^k_{\mathbb{C}}$-bundle over some Kähler manifold W^0. Let π denote the projection to W^0.

The space of $\mathbf{T}^k$-invariant Kähler potentials is denoted by $\mathcal{H}_{\mathbf{T}^k}(M,\omega)$ while the space of $\mathbf{T}^k$-invariant ω-psh functions is denoted by $PSH_{\mathbf{T}^k}(M,\omega)$.

2.2. **Local potentials.** Proposition 6.8 states that there is a covering $\{U_i\}$ of W^0 such that ω has a $\mathbf{T}^k$-invariant potential u_i on each $\pi^{-1}(U_i)$. If w_j are local holomorphic coordinates on U_i and $z_j = x_j + iy_j$ are coordinates on $\mathbf{T}^k_{\mathbb{C}} = (\mathbb{C}/i\mathbb{Z})^n$ then $u_i(x,y,w)$ is independent of y and convex in x. Moreover the momentum map $\mathbf{m}$ is locally given by $\nabla_x u_i$. This is stated in Proposition 6.9.

Given $\phi \in \mathcal{H}_{\mathbf{T}^k}(M,\omega)$ then clearly $u_{\phi,i} := u_i + \phi$ is a local potential for ω_ϕ, and the corresponding momentum map of (M,ω_ϕ) is of course given by $\mathbf{m}_\phi = \nabla_x u_{\phi,i}$.

2.3. **The partial Legendre transform.** Recall that if $u(x)$ is a convex function on $\mathbb{R}^k$ then its Legendre transform $u^*(p)$ is the convex function on $\mathbb{R}^k$ defined as

$$u^*(p) := \sup_{x\in\mathbb{R}^k} \{\langle p,x\rangle - u(x)\}.$$

The Legendre transform plays a key role in convex analysis, as explained in Section 2 of the thesis.

Thus given the local potential $u_i(x,y,w)$ (or more generally $u_{\phi,i}$) one can perform the Legendre transform in the x-variables:

$$u_i^*(p,w) := \sup_{x\in\mathbb{R}^k} \{\langle p,x\rangle - u(x,y,w)\}.$$

This is then convex in p, and Kiselman's minimum princle [Kis78] also gives that it is plurisuperharmonic in w.

2.4. **Symplectic reduction revisited.** Let p be a regular value of the momentum map $\mathbf{m}$. We saw above that symplectic reduction yields a symplectic manifold (M_p,σ_p). It is easy to see that one locally can identify M_p with W^0, giving it a complex structure. The key result Theorem 6.10 now states that locally

$$\sigma_p(w) = -i\partial\bar{\partial}u_i^*(p,w),$$

i.e. $-u_i^*(p,w)$ is a local Kähler potential for σ_p.

2.5. **Reduction currents in more singular cases.** Since the partial Legendre transform is robust and allows singularities, Myga can define the reduction current $\sigma_{\phi,p} := -i\partial\bar{\partial}u^*_{\phi,i}(p,w)$ for any $\phi \in PSH_{\mathbf{T}^k}(M,\omega)$.

In Proposition 6.11 and Theorem 6.12 shows that the operation of taking the reduction form or some power of it is continuous under decreasing limits of potentials in $\mathcal{E}_{\mathbf{T}^k}(M,\omega)$, and also continuous in p.

2.6. **Density of the volume form.** Note that the local density of the volume form ω^n with respect to the local volume form $dx \wedge dy \wedge dw \wedge d\bar{w}$ is given by $\det D^2 u_i(x, y, w)$, where D^2 denotes the complex Hessian. Theorem 6.14 gives the following explicit formula for this density:

$$\det D^2 u_i(x, y, w) = (-1)^{n-k} \det D^2_w u_i^*(\nabla_x u(x, y, w), w) \det D^2_x u(x, y, w).$$

Here D^2_w denotes the complex Hessian in the variables w while D^2_x denotes the real Hessian in the variables x. In the proof Myga explicitly computes the different determinants, expertly finding all cancellations.

3. Complex optimal transport and the (complex) g-Monge-Ampère equation

Motivated by the study of Kähler-Ricci solitons Berman and I introduced a g-Monge-Ampère equation in the Kähler setting [BWN14].

3.1. **g-Monge-Ampère measure.** Let as before (M, ω) be a compact Kähler manifold with a Hamiltonian $\mathbf{T}^k$-action. Let ϕ be a $\mathbf{T}^k$-invariant Kähler potential and let g be a positive continuous function on the momentum polytope Δ. Then

$$MA_g(\phi) := g(\mathbf{m}_\phi) MA_\omega(\phi)$$

defines a measure on M called the g-Monge-Ampère measure of ϕ. However, at the outset it is less clear how to extend this definition for more general ϕ, given that the moment map $\mathbf{m}_\phi$ might not be properly defined everywhere.

In Section 7 Myga provides a definition of the g-Monge-Ampère operator mostly following that in [BWN14], but with more details. He also establishes its key continuity properties, such as Theorem 7.5, which states that if $\phi_j \in \mathcal{E}_{\mathbf{T}^k}(M, \omega)$ either decreases to a full mass potential ϕ or increases almost everywhere to ϕ, then

$$MA_g(\phi_j) \to MA_g(\phi)$$

weakly.

3.2. **g-Monge-Ampère equation.** Let μ be a $\mathbf{T}^k$-invariant measure on M which puts no mass on pluripolar sets and such that $\int_M \mu = \int_M \omega^n$. Let g be a positive continuous function on Δ. We are looking to solve g-Monge-Ampère equation

$$MA_g(\phi) = \mu.$$

As said above the motivation for considering this equation comes from the study of Kähler-Ricci solitons, and is described briefly in Section 8.1 of the thesis.

In Section 8.2 Myga proves the existence of a solution in the toric case, i.e. $k = n$. The idea is to reduce it to the real optimal transport problem, and the theorem of McCann. The issue is that McCann's theorem assumes that μ puts no mass on sets with Hausdorff dimension less than or equal to $n-1$, so in particular point masses are not allowed. However, a point mass in $\mathbb{R}^n$ corresponds to a measure with support on a torus in M, and so should be allowed in the complex setting. Myga solves this by considering the dual transport problem, i.e. instead of transporting μ to ν we transport ν to μ. The point is that $\nu = g(p)dp$ is assumed to be nice, so by McCann's theorem there is a solution to the dual problem. Now, let v denote the convex function whose subgradient solves the dual transport problem. If v happened to be smooth and strictly convex then the gradient of v would be the inverse of the gradient of the Legendre transform v^*, i.e. v^* would solve the original problem. In fact, by approximating v by smooth and strictly convex functions v_n Myga proves that v^* does indeed induce an unique solution to the (complex) g-Monge-Ampère equation. For this he uses various quite subtle convergence properties of convex functions and their Legendre transforms established in Section 2.

In Section 8.3 Myga solves the g-Monge-Ampère equation on general manifolds using instead a variational method, mostly following [BWN14], but again with more detail. He also use the formula for the density of Monge-Ampère measures proven in Section 6 to prove uniqueness (up to constants) given that the momentum maps are equal (Corollary 8.12).

4. Comments and questions

4.1. **p. 70.** Is the first part of the argument on page 70 using the coisotropic embedding theorem a proof of Theorem 5.27? Could this argument be used to calculate the density of the volume form ω^n?

4.2. **p. 77.** I would have liked it to be more clearly said that $u^*_\psi(p,w)$ is the partial Legendre transform introduced by Kiselman, and that the plurisuperharmonicity of $u^*_\psi(p,w)$ in w is known as Kiselman's minimum principle.

4.3. **p. 78.** Does the plurisuperharmonicity of u^*_ψ follow from Theorem 6.10 (by approximation)? Is the proof of Theorem 6.10 related to Kiselman's proof of his minimum principle?

I guess there are descriptions of Kähler potentials for the Kähler form on the reduced space in the literature, e.g. [BG04] and references therein. How does Theorem 6.10 relate to those?

4.4. **p. 80.** Is this definition new or does it exist in the literature? If p is not a regular point of m_ϕ, what kind of singularities can one expect of $u^*_\phi(p,w)$ and $\sigma_\phi(p,w)$? Are there easy examples?

4.5. **p. 82.** If one formulates the formula of Theorem 6.14 as a relationship between densities of measures on M, the reduced space M_{red} and $\mathbb{R}^n$, is it then true in the general symplectic setting, or does it rely on the Kähler structure?

Integrating along the fibers of the moment map, do we then get a new proof of Duistermaat-Heckman's theorem?

4.6. **p. 86.** I believe that the envelopes used here in the construction of the g-Monge-Ampère measure are special cases of those used in [BWN14]. And I don't think you can get all step functions from using a particular lexicographic order. But maybe you are not really thinking of the lexicographic order? Anyway, you should be able to get envelopes from subsets of Δ, so it is not a serious issue.

4.7. **p. 88.** To me the proof that $MA(P_q\phi)$ is supported on $\{P_q\phi = \phi\}$ seems more complicated than that in [BWN14]. Was there a specific reason for doing it your way?

4.8. **p. 108.** A more naive approach to Prop. 8.11 and Cor. 8.12 would be to say that if u, v are smooth, $m_u = m_v$ and $MA_g(u) = MA_g(v)$ then $MA(u) = MA(v)$, and hence $u = v + C$, and then use some approximation argument. Does this also work or are there issues?

Does thinking of the g-Monge-Ampère equation as a family of optimal transport problems on the fibers of the $\mathbf{T}^k_{\mathbb{C}}$-action lead to an alternative definition of the g-Monge-Ampère operator, and if so could it be used to define it for more general g than the positive continuous ones considered here and in [BWN14]?

5. OPINIONS

The preliminary parts of the thesis gives an excellent introduction to the topic of torus actions on symplectic and Kähler manifolds. It is well-structured, clearly written with lucid arguments. Thanks to this I learned a lot from reading it.

In the thesis Myga shows great technical skill, mastering methods of Lie group theory, symplectic geometry and pluripotential theory.

The connection between Kiselman's partial Legendre transform and symplectic reduction exhibited in Theorem 6.10 is striking, and the formula relating the densities in Theorem 6.14 is very pretty. I have personally used the partial Legendre transform in several papers, and I believe this formula could have been used to prove some of my results.

The paper [BWN14] has a lot of citations, but for various reasons was never published, and it also suffers from a lack of detail in several places. I am therefore happy to see Myga's nice alternative proof of existence in the toric case, as well as his detailed exposition of the variational proof in the general case. The new approach of Myga looking at it as a family of optimal transport problems on the fibers of the action is also exciting.

In conclusion I am impressed by the thesis and of course recommend that Myga be awarded a doctorate on the basis of it.

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