Review of the PhD dissertation "Medial axis and singularities" by Adam Białożyt

The thesis deals with the medial axis of a closed set. Let me recall the definition of the medial axis and the reasons why it is an object which naturally appears in different contexts. (In the sequel we adopt the notation used in the dissertation, even numbering and references refer to the ones of the thesis.)

Given a closed set, $X \subset \mathbb{R}^n$, and a point $a \in \mathbb{R}^n$, we define the metric projection of a into X as

$$m(a) = \{x \in X \,:\, \|a - x\| = d(a, X)\},$$

where $\|\cdot\|$ is the Euclidean norm and $d(a, X)$ stands for the Euclidean distance of a from X. Then, the medial axis of X, M_X, is the set of all the points $a \in \mathbb{R}^n$ so that $m(a)$ is not a singleton, i.e. the set of all the points where the metric projection is a multivalued map.

The medial axis naturally appears both in pure and in applied mathematics, indeed:

- if $a \in M_X$ then there is a geodesic, starting at a point of X, which ceases to be minimizing at a (in other words M_X is contained in the cut locus of X);
- in a pde's perspective, the distance function is the unique (viscosity) solution of the homogeneous Dirichlet problem for a constant coefficients, (stationary) eikonal equation and M_X is the set where such a function is not differentiable (we observe that $\overline{M_X}$ is the C^1 singular support of the distance function);
- from the point of view of the mathematical control theory, the elements of the medial axis are the starting points of several optimal trajectories for the simplest minimum time problem.

Further motivations for the study of the medial axis, probably, come from applied mathematics (shape recognition, ...), but this is out of my expertise.

Some results on the structure of the medial axis can be derived as a consequence of the regularity of the distance function: since the Euclidean distance is a viscosity solution of an Hamilton-Jacobi equation with coercive and strictly convex Hamiltonian, then it is a semiconcave function (i.e. it can be locally written as the sum of a concave with a smooth function). In this perspective, it is well-known that the medial axis is a set of measure zero and its Hausdorff dimension is at most

$n - 1$ (it easy to see that this bound is optimal considering elementary examples such as a square in the Euclidean plane). Furthermore, even lower estimates on the Hausdorff dimension of the medial axis are available[1]. Unfortunately, M_X is not a closed set and, in particular, it is "unstable".

In the thesis under review, the author takes a very interesting perspective: in order to prove precise results on the structure of the medial axis, he considers X in a subclass of the closed subsets of $\mathbb{R}^n$ (i.e. the closed sets definable in o-minimal structure extending the field of the real numbers).

The first problem considered in the thesis is the study of the (Peano) tangent cone at $a \in \overline{M_X}$, $C_a M_X$. More precisely, for $X \subset \mathbb{R}^n$ closed a definable set, given point $a \in \overline{M_X}$ (up to a translation, one can assume that $a = 0$), then

$$M_{m(0)} \subset C_0 M_X. \qquad (A)$$

(This is the content of Theorem 4.6.) Let me point out that the ingenious proof of this result is based on the analysis of the graph of the distance function and, implicitly, on the curve selection lemma (which holds in the o-minimal setting). A more precise result holds in the case of $n = 2$ (the inclusion in (A) is an equality). A direct consequence of Theorem 4.6. is the lower bound

$$\dim C_a X \geq \dim M_{m(a)}, \qquad a \in M_X \qquad (B)$$

given in Corollary 4.11.

The second problem considered in the thesis is the study of the dimension of the medial axis. The main result is given in Theorem 4.21: if $X \subset \mathbb{R}^n$ is closed and definable, then

$$\dim_a M_X + \min\{k \,:\, a \in \overline{M^k}\} = n - 1, \qquad (C)$$

for every $a \in M_X$, where $M^k = \{a \in M_X \,:\, \dim m(a) = k\}$.

Let me remark that, from a technical point of view, the proof of Theorem 4.21 is based on the existence of a cylindrical definable cell decomposition (cdcd) adapted to a finite number of definable subsets of $\mathbb{R}^n$ (the existence of this decomposition is guaranteed by the assumption that the set X is definable).

The third problem considered in the thesis is related with the notion of reaching radius. It is well-known that, if ∂X is less than C^2, some

[1] In the special case of the Euclidean distance function, if $a \in M_X$ and ν is the dimension of the normal cone to the convex hull of $m(a)$, co $(m(a))$, at a point in ∂ co $(m(a)) \setminus m(a)$, where ∂A stands for the topological boundary of A, then M_X has positive ν-dimensional density at a.

"pathological" phenomena can occur. For instance, while, as a consequence of the Rademacher theorem on the a.e. differentiability of the Lipschitz functions, M_X is of measure zero its closure, $\overline{M_X}$, may be of positive measure. Then, it is interesting to detect the points in $\overline{M_X} \setminus M_X$. This is done in Theorem 4.31 by means of the notion of limiting directional reaching radius. We point out that in such a result X is not assumed to be definable. We observe that, in the thesis, a suitable notion of reaching radius is also introduced and it is compared with analogous concepts introduced by other authors.

In Theorem 4.46 it is given a sufficient condition ensuring that a point which is in the set of the C^2 (but not C^1) singularities of the set X belongs to the closure of the medial axis. Finally, also some results on the C^1 singular case are provided.

Let me make some comment and suggestions in order to improve the thesis.

Suggestions.

(1) For the readability of the text the definitions 3.44, 4.16, 4.28 and 4.41 should be illustrated by examples.
(2) In the proof of Theorem 4.6 a key role is played by the assumption X definable. Such an assumption is used in Corollary 3.32 which is not proved in the present thesis. Either a complete or at least a sketch of the proof of Corollary 3.32 should be provided.
(3) The relationship (if any) of Theorem 4.6 and Corollary 4.10 with the existing results on the subject should be put in evidence. More precisely, formulas (B) and (C) above should be compared with lower bound estimates on the Hausdorff dimension of the singular set given in the references [ACS93] and [AC99].
(4) I think that a discussion of the problems addressed in the thesis, restricted to the semialgebraic setting, could be of great interest. For instance, can one get the same results or, at some point is one forced to study the problems in a more general setting?

Minor remarks. Even if I found very useful the list of the symbols at the end of the thesis, I think that in order to save the reader a lot of page flipping, it would be better to define the symbols in text right before their first occurrence.

Misprints and minor comments

- page 13 line -12 (from the bottom) it should be "neighbourhoods of x" instead of "neighbourhoods of t";
- page 14 line 3 the word "definable" is used but it is not previously introduced;

- page 14 line -4 the "natural projection" should be defined;
- page 15, in Example 3.4. point 3. the symbol $\mathbb{P}_1^n$ is used but it is not previously introduced (stands for the n-dimensional projective space?);
- a precise reference for the proofs of the theorems 3.5, 3.6 3.8 and 3.12 and of the corollaries 3.7 and 3.9 should be given;
- it is difficult to find the definition of dimension used in the manuscript (which by the way is relevant also for suggestion (3) above): it is given in page 17, right after Theorem 3.12. Since it seems to be strictly related to the notion of definable, I think that more emphasis and some explicit examples are needed;
- page 32 line -6 it should be

$$\left(\frac{d(a)+p^{(n+1)}}{2d(a)}a+\frac{d(a)-p^{(n+1)}}{2d(a)}m(a)\right)\times\left\{\frac{d(a)+p^{(n+1)}}{2}\right\}$$

instead of

$$\frac{d(a)-p^{(n+1)}}{2d(a)}m(a)\times\left\{\frac{d(a)+p^{(n+1)}}{2}\right\};$$

- page 41, line 16 it should be "Theorem 4.20" instead of "Corollary 4.20".

Concluding remarks. The thesis is clearly and carefully written and I think that it is a serious work of research on basic phenomena. The problems considered are addressed by means of highly non-trivial geometrical techniques. I strongly recommend that the thesis be accepted by the committee.

Paolo Albano — February 21, 2022

Paolo Albano — Date