UNIMORE
UNIVERSITÀ DEGLI STUDI DI
MODENA E REGGIO EMILIA

Dipartimento di Scienze
e Metodi dell'Ingegneria

Sede
Via Giovanni Amendola, 2
42122 - Reggio Emilia, Italia
T +39 0522 52 2161

www.unimore.it
www.dismi.unimore.it

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Review of the PhD thesis of Mr. Michael Lipiński

Dear ladies and gentlemen,

It is my pleasure to present the following review of the PhD thesis of Mr. Michael Lipiński entitled "Morse-Conley-Forman theory for generalized combinatorial multivector fields on finite topological spaces".

1. Classification and contributions

In his thesis, Mr. Lipiński works on some theoretical problems concerning the theory of discrete multivector fields, with a focus on the analogues of classical Morse and Conley theories.

Forman's theory concerning discrete vector fields is a highly relevant research topic in applied and computational topology and aims at the extension of smooth vector fields concepts to discrete data as occur in many applied sciences including geology, geography, and engineering. The goal of topological approaches to vector fields analysis is to learn qualitative properties about dynamical systems useful especially when analytical solutions are not available. In application it is often the case that even the continuous vector field is unknown, and only a sample of it is available. The challenges in this situation are manifold. To name a few,

- how to construct a discrete vector field given a vector cloud?
- how to define index pairs, Conley index, limit sets, attractors and repellers, Morse decompositions?
- how does noise and parameter choices influence these constructions?

As for the first two questions, the state-of-the-art results in the literature before those of this thesis are due to Marian Mrozek in [21], who presented the idea of using multivectors rather than vectors. As for the third question, the reference paper is [8] where zigzag persistence is used to summarize changes to the Conley index of a multivector field in the presence of noise.

In this context, Mr. Lipiński makes the following contributions to the construction of a combinatorial analogue of the discrete theory of topological dynamics:

- Defines multivector fields defined on more general domains, extending from simplicial complexes to finite topological spaces;
- Eases the assumptions for multivector fields necessary.

On the one hand these changes require new proofs, on the other hand are claimed to lead to a theory more convenient for applications.

Mr. Lipiński's thesis is based on the published papers [17] and [7] of which he is one of the authors. It is written in English, circa one hundred pages long, divided into six chapters after a brief introduction, and terminates with a short table of symbols, an index of terms, and a list of references.

2. Chapters' overview

The **introduction** informally presents in two pages the motivations for introducing multivectors in the combinatorial approaches to dynamics. For that it is noticed that they allow to overcome limitations of the Forman's theory. The thesis contributions are then listed in one small paragraph which amount to removal of limitations of the previously existing theory of multivectors. The introduction ends with a detail of the structure of the document and a declaration about authorship of results.

Apart from the mathematical interest of lifting results to more general settings, it is said, without too much explanation, that there are computational motivations behind this choice. Elaborating more in the introduction or in a separate chapter about the computational benefits of lifting those assumptions would certainly increase the interest of the reader.

Chapter 1 introduces very basic preliminaries and sets the notations used throughout the thesis. The covered topics are sets and maps, relations and digraphs, orders and posets, topological spaces, elementary algebraic structures such as groups, rings and vector spaces.

This preamble makes the thesis self-contained. However, as these notions are very elementary, it serves mainly to fix notations. In this respect, it must be noticed that some of the symbols introduced (e.g., that of multivalued map) are not listed in the index of symbols. This remark extends to other symbols introduced later in this work.

As for the choice of symbols, one that seems unclear to me is \subset versus \subseteq to indicate the strong versus the weak subset relation (cf. also the symbol \subsetneq on page 14 and the last two lines of page 17). For example, I believe that the Proof of 1.4.7 uses \subset to mean the weak subset relation, otherwise it does not work.

The term "number" denoted by # could be made more precise using the term "cardinality" (useless if sets are assumed to be finite).

Minor comments:

P. 8, l. 10: add that A is not equal to A'
Caption of Fig. 1.1: Directed digraph → directed graph, (C,B) → (B,C), (D,C)→(D,B)
P. 10, l. 8: Full stop is missing
P. 11, (T3): The union is taken over a wrong set
P. 11, l. -8: satisfies.
The proof of Prop. 1.4.10 works but one should not start assuming that y is in A.
P. 15: It is important to say that the neutral element is denoted by 0 and the inverse of a is denoted -a. Moreover, one needs (-n)a=n(-a) to pass from n in the natural numbers to n in the integers.

Chapter 2 reviews the needed background on algebraic topology. It covers simplicial complexes (both geometric and abstract), order complexes, simplicial homology, chain homology, singular homology. The latter is necessary as the thesis deals with finite topological spaces.an McCord theorem bridging it to simplicial homology is also reviewed. The chapter end with a short review of zigzag persistence.

The chapter is very well organized and rich with very useful explanations, both verbal and pictorial. The writing is mostly accurate, apart from the fact that in defining submodule and interval modules of zigzag persistence it is important to define the internal maps, not only the vector space.

I am not sure I correctly understand Figure 3.8, as it refers to the zigzag of figure 2.7 that is not exactly as written in (2.12) where we have also K_{5\cap 6} and K_{6\cap 7}. I wonder if these need to be considered as steps of the filtrations which therefore should vary from 0 to 11 rather than 9.

I also believe there are a number of typos:

P. 18: Geometric or geometrical?
P. 19: an n-skeleton → the n-skeleton
P. 20: of a simplicial complexes → of a simplicial complexes; regular simplicial complex → geometric simplicial complex; the symbol K in the displayed formula on line 15 should not be in calligraphic font.
P. 21 l. -3: X and Y need to be finite
P. 23: its subcomplex → a subcomplex of it
P. 24, l. 2: chain complexes → simplicial complexes
P. 25: need to say that the ith coordinate of e_i is equal to 1
P. 27: that connecting; convection;
P. 28: I guess the edges in the graphs representing the topological spaces in fig. 2.4, 2.5 and 2.6 need arrows and \delta(x,y,z) in fig. 2.4 should rather be \delta(x,y,z,0,0,…)
P. 31: a zigzag persistence

and some clashes of notations:
P 22: the subscript of a chain c takes two different meanings.
P. 24: same for \delta

Chapter 3 is also a review of preliminary notions, this time from the theory of dynamical systems. It starts with the theory in the continuum before passing to the combinatorial one. For combinatorial systems, the evolution of the theory is presented starting with Forman's approach before passing to Mrozek's and finally the one adopted in this work. What is nice is that all these approaches can be viewed as examples of a general definition of combinatorial dynamical system. Unfortunately, an explanation about why Mrozek's approach may be preferable to Forman's one is missing. On the contrary, an example showing why the approach of this work solves some limitations with Mrozek's approach is given in Figure 3.8. Maybe giving more prominence to it would be useful. However, one must wait until the following chapter before realizing that also some definitions become simpler.

Again, the chapter is well organized and mostly well written. Overall, it demonstrates that Mr. Lipiński has a comprehensive understanding of his a
rea of research.

Some minor comments are:

P.34: - A solution of a point x ∈ X, is the map → The solution at a point x ∈ X is the map;
- with corresponding isolated neighborhoods → with corresponding isolating neighborhoods;
- in (2) use P_1 instead of N?

Figure 3.1 and 3.4: Why showing the trivial summands? If they are useful for explanatory reasons it is not evident.

P. 35: An α- and ω-limit sets of x → The α- and ω-limit sets of x as I believe they are unique.

P. 36: I think that discussing if Morse decompositions are unique would be useful.

P. 38, l. -8 and -9: t → i

P. 39: in a simplicial complexes,

P. 40, l. -4: Is it correct to use simplicial homology here?

Figure 3.8: Starting with this figure, a new way of representing multivectors is adopted. Some explanation to the reader would be useful.

Chapter 4 contains the core results of the thesis, namely a modified theory of combinatorial multivector fields is introduced and its Morse and Conley properties are investigated. The main theorems in my opinion are Thm. 4.2.5, which justifies the definition of Conley index for the considered multivector fields, Thm. 4.2.9 that shows the additivity of the Conley index, Thms. 4.3.3 and 4.3.4, which characterize attractors and repellers, Thm. 4.422, which states Morse inequalities.

These results are very interesting and convincingly show that the theory is well constructed. For example, the definition of a critical multivector turns out to be much simpler than in that of [21]. This kind of comparative improvements is missing while I believe a detailed comparison would help the reader.

The chapter would have benefitted from more verbal explanations as they are very limited: Often one simply reads a list of formal results. However, I have appreciated the wealth of illustrations.

What is not clear to me is why some results, not only propositions but even theorems, are stated without proof. I think this could be clarified in the last paragraph of the introduction. However, this adds to the feeling of reading a list of formal statements without verbal comments helping the comprehension.

Minor comments:

Figures 4.2, 4.3: is it correct to use calligraphic K in the caption as the order complex corresponds to the barycentric subdivision?

P. 50: I missed the definition of periodic solutions.

P.51: Amend the sentence “ the invariant part of a dynamical system to an invariant set needs not be invariant”

Figure 4.7: The content of the middle picture is not commented in the caption. Where is P_1?

P. 64: Inv A = 0 → Inv A = \emptyset

P. 66: Does Prop. 4.3.1 hold also for the empty set?

P. 68: opn x ∈ A* → opn x ⊂ A*
which proves that X ⊂ A* → which proves that opn x ⊂ A*

P. 74: The paragraph about the example of Figure 4.9 should follow the proof of Thm. 4.3.22. Mind that in the text capital letters are used for vertices contrary to what is shown in the figure itself.

Chapter 5 introduces the study of persistence of Morse decomposition. The introduction of the chapter does not spend words on the questions motivating this study nor on the obtained results. It is mentioned that the purpose is studying the evolution of Morse sets. The tool used to achieve this is the comparison diagram. The difficulties are: first Morse sets are not disconnected, thus motivating the use of the disconnecting topology and weak Morse sets; secondly, weak Morse decompositions are not unique and can contain sets too small to be meaningful, thus motivating the use of strongly connected components and minimal weak Morse decompositions. Here one can remark that it would be useful for the reader to explicitly underline these connections between Section 5.2 and Section 4.4.2.

What is meant by evolution of Morse sets is not explained, but, as far as I understand after reading Chapter 6, one wants to compare discrete multivector fields sampled under different noise intensities or different choices of sampling parameters. A few words anticipating this ideas would be useful at this point.

Again, many results (e.g. Thm 5.1.1) are stated without a proof.

There is a typo in the caption of Fig. 5.1: {c}→{e}

Chapter 6 presents the developed theory at work on numerical examples. To do this some algorithms are presented. The first algorithm, called CVCMF and whose correctness is proven, constructs a multivector field starting from a sampling of a vector field on a simplicial complex. The second algorithm, MC-graph, computes weak Morse decompositions. The third algorithm, MDPersistence, sets up the input data for an external routine that computes zigzag persistence. Finally, this algorithmic pipeline is tested to check the influence of noise or change of sampling parameters.

Numerical results are very encouraging that the theory of discrete multivector fields can be very useful to study dynamical systems from a sampling of them.

It must be noticed that algorithm CVCMF does not exploit the full generality allowed by the theory developed in this thesis as it constructs a multivector field according to [21]. So, testing the full generality allowed by the results of this thesis is left to future research. Moreover, CVCMF is different from CMVF given in [21] but the differences are not commented, nor the outputs compared.

Algorithm MC-graph is not explicitly given but verbally described. It requires extracting strongly connected components of a graph. It would be useful to give a reference to the literature for ways to achieve this.

Apart from this, the chapter is very well presented and convincing.

The **Bibliography** contains the essential references for this work. I appreciate it is not needlessly overcrouded.

In reference [19], conley needs capital initial.

3. Recommendation

Contents: This work makes an important improvement in the developing of the theory of discrete multivector field. Overall, the thesis is well structured and the research agenda is clear. The results are original, and proofs are mathematically sound.

Knowledge: Mr. Lipiński demonstrates a profound understanding of the techniques from [21]. The discussion of possible applications and extensions demonstrate Mr. Lipiński expertise in the wider area of topological persistence.

Presentation: The thesis is well structured and the line of thought is immediately understandable. What could be further improved is a more extensive use of verbal motivations in Chapters 4 and 5 and more detailed comparisons with the state of the art.

Recommendation: In summary, the work is original, well designed, comprehensive, and appropriately presented. Hence, I recommend awarding the degree to the candidate:

Yes, except for minor amendments as listed above

Following the argumentation above, the overall **evaluation** is

VERY GOOD, i.e. very good and solid work, definitely meritorious.

Sincerely,

Claudia Landi

Prof. Dr. Claudia Landi