Referee's report for the Habilitation of Dr. Maciej Piotr Denkowski

The material presented by Dr. Denkowski as the basis for the Habilitation procedure consists of 6 papers [A1–A6]. (The citations in my report are the same as in the 'Summary of Professional Accomplishments' provided by the candidate.)

The first paper [A1] deals with the Kuratowski convergence of complex analytic sets. (The Kuratowski convergence is a natural extension to closed sets of the Hausdorff convergence of nonempty compact sets.) In general, the Kuratowski limit of a convergent sequence of analytic sets need not be analytic. A sufficient condition for the limit to be analytic is given by the famous Bishop theorem. Tworzewski and Winiarski gave a version of this theorem for algebraic sets. In Theorem 3.4 of [A1], Denkowski and Pierschała prove a version of these results for Nash sets, providing a sufficient condition for the limit of a convergent sequence of Nash sets to be a Nash set too. The condition in question involves the notion of degree of Nash sets, which is defined, in quite a natural way, in the paper. Conditions for the limit to be algebraic are given as well.

As an important corollary of Theorem 3.4, the authors obtain a version for Nash sets of the Chevalley-Remmert projection theorem. Namely, they show that the closure, in an open set Ω of $\mathbb{C}^n$, of the first projection of a Nash subset of $\Omega \times \mathbb{C}^m$, with finitely many irreducible components, is a Nash set in Ω (Corollary 3.5).

In Theorem 3.7 of the paper, they also provide a sufficient condition for a purely k-dimensional analytic set $A \subseteq \mathbb{C}^n = \bigcup U_i$ (where $U_1 \subseteq U_2 \subseteq \cdots$ is an increasing sequence of open subsets) such that each $A \cap U_i$ is the Kuratowski limit of a convergent sequence of purely k-dimensional analytic subsets of U_i, to be algebraic. For that purpose, they introduce a new notion of global degree of an analytic set.

In the second paper [A2], Z. Denkowska and M. Denkowski investigate the Kuratowski convergence of the sections of a closed definable set. Here, 'definable' means definable in some o-minimal structure expanding the field of real numbers. It is worth mentioning that, in the definable setting, almost nothing was written in this domain. The paper [A2] — which therefore may be viewed as a pioneer — contains many very interesting results. It starts with basic properties of the Kuratowski convergence in this setting, like for instance, the definability of the upper and lower limits (Theorem 2.5) or the 'almost everywhere' (in the definable sense) continuity of the sections (Theorem 2.11).

In Theorems 3.7 and 3.9, the authors investigate the behaviour of the connected components of the sections of a compact definable set under the operation of taking the Kuratowski limit of the sections. In Theorem 3.1, they show the definability of the function counting the number of connected components of the section, and in Lemma 3.10 and Theorem 3.13, the definability and the semicontinuity of the function giving the dimension of the section.

Subanalytic versions of these results are presented as well.

As an application, the authors obtain, in Theorem 4.1, a result on semi-algebraic approximation of compact subanalytic sets (in the same spirit as Bliski's work in complex analytic geometry).

The starting point of the paper [A3] is a result of Nash saying that any analytic submanifold M of $\mathbb{R}^n$ has an arbitrarily small neighbourhood in which each point x has a unique closest point $m(x) \in M$; the result also asserts that the function $m(x)$ is analytic. Then we may ask what happens if we allow M to have singularities, and in this case, what is the structure of the exceptional set (called 'medial axis') of points for which there is more than one closest point. The main result of this paper (Theorem 2.1) deals with these questions in the definable setting with parameters. Precisely, it is shown that for any definable set $M \subseteq \mathbb{R}^k_t \times \mathbb{R}^n_x$ with locally closed t-sections M_t, there exists a definable set $W \subseteq \mathbb{R}^k_t \times \mathbb{R}^n_x$ with open t-sections W_t such that $M_t \subseteq W_t$ is closed in W_t and $m(t,x) \neq \emptyset$ for $x \in W_t$, where $m(t,x) := \{y \in M_t\,;\, \|x-y\| = \operatorname{dist}(x, M_t)\}$, $(t,x) \in W$. Moreover, the multifunction $m(t,x)$ is definable; there is a definable set $E \subseteq W$ with nowhere dense sections such that $\#m(t,x) = 1$ iff $x \in W_t \setminus E_t$; and for any integer $p \geq 2$, there is a definable set $F^p \subseteq W$ containing E with closed and nowhere dense sections F^p_t such that M_t is a $\mathscr{C}^p$ submanifold near $x \notin F^p_t$ and $m(t,\cdot)$ is $\mathscr{C}^{p-1}$ near $x \in W_t \setminus \overline{E_t}$ iff $x \notin F^p_t$.

A subanalytic counterpart of this result, without parameters, is also given in Theorem 3.2. In the last part of the article, the author discusses the properties of the corresponding multifunction $m(x)$.

The paper [A4] is a continuation of [A3]. There, Birbrair and Denkowski investigate the medial axes of sets definable in polynomially bounded o-minimal structures. The paper contains many very interesting results and proofs, which can be organized into two parts. In the first one (Section 2), the authors present general basic results about the medial axis. For instance, in Theorem 2.23, an elementary proof of the characterization of the medial axis as the set of non-differentiability points of the square of the distance function (to the given closed definable set) is given. The second and main part (Sections 3 and 4) deals with, on one hand, the problem of characterizing the points of a closed definable set which are reached by the medial axis, and on the other hand, the question of computing the Peano tangent cone of the medial axis at such a point. These questions are thoroughly solved in the special case of a definable curve in $\mathbb{R}^2$, using a notion called 'superquadraticity'. The corresponding theorems in the text are 3.19, 3.21, 3.24 and 3.27. The general case is discussed in Section 4 using a notion called 'reaching radius'. See Theorems 4.6 and 4.35.

The motivation for the paper [A5] comes from a classical result — which goes back to Remmert — characterizing open holomorphic maps of $\mathbb{C}^n$. It says that a holomorphic map between domains of $\mathbb{C}^n$ is open if and only if its fibres are discrete. A generalization of this result to the real setting was given by Gamboa and Ronga for polynomial maps, and by Hirsch in the analytic case. In Theorem 3.14 of [A5], Denkowski and Loeb give a $\mathscr{C}^1$ subanalytic/definable version of this result. The proof is based on a characterization of open maps in terms of the continuity of its fibres with respect to the Kuratowski convergence (proved in Lemma 3.8).

In the second part of the paper the authors discuss the relation between openness and properness in the case of analytic mappings between domains of $\mathbb{R}^2$ (Theorem 4.5).

The last article [A6] takes us back to complex analytic geometry. For an irreducible, locally analytic set $A \subseteq \mathbb{C}^m \times \mathbb{C}^n$ with $p^{-1}(0) \cap A = \{0\}$, where p is the second projection, the main result of this paper (Theorem 2.3) gives the intersection multiplicity $i(p^{-1}(0) \cdot A; 0)$ — in the sense of Achilles-Tworzewski-Winiarski — as the product of the local degree (Lelong number) $\deg_0 p(A)$ and the so-called 'regular' multiplicity $\tilde{m}_0(p_{|A})$. This generalizes a similar formula of Spodzieja, which corresponds to the special case where A is the graph of a holomorphic map $f\colon U \to \mathbb{C}^n$ defined on a neighbourhood U of $0 \in \mathbb{C}^m$.

As an application, Denkowski gives a new simple proof of a theorem of Ebenfelt and Rothschild, which says that if $F\colon (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ is the germ of a finite holomorphic mapping and $V \subseteq \mathbb{C}^m$ a complex analytic set germ at 0 such that $V = F^{-1}(F(V))$ and V is smooth, then, provided $\operatorname{Jac} F_{|V} \not\equiv 0$, the image $F(V)$ is smooth too.

Besides the series of articles [A1–A6], which serves as a basis for the Habilitation procedure, Dr. Denskowski is also the author of another collection of works — presented as 'Other Scientific Achievements' in his 'Summary of Professional Accomplishments' — consisting of 10 papers [B1–B5, C1–C5], a book chapter [D1], and 7 preprints [P1–P7] currently submitted for publication.

In the group of articles [B1, B2, B4, C2], Denkowski investigates properties of c-holomorphic mappings (i.e., continuous mappings $f: A \to \mathbb{C}^n$, where A is an analytic subset of an open set $\Omega \subseteq \mathbb{C}^m$, such that $f_{|\operatorname{Reg} A}$ is holomorphic). A special emphasis is placed on numerical biholomorphic invariants of such mappings. For instance, in Theorem 5.5 of [B1], generalizing a holomorphic result of Płoski and Chądzyński, an estimate of the Łojasiewicz exponent of a c-holomorphic mapping, in the case of an isolated zero, is given.

Another important exponent is the Hilbert's Nullstellensatz exponent. In Theorem 3.2 of [B2], Denkowski proves an effective Nullstellensatz for c-holomorphic functions in the case of a proper intersection. This result is extended to the case of an isolated improper intersection in Theorem 4.1.

In the paper [B4], Denkowski extends to the c-holomorphic setting the Coleff-Herrera notion of residue current. In particular, a c-holomorphic counterpart of the Lelong-Poincaré formula is given.

With [B3], we come back to Łojasiewicz inequality. The Łojasiewicz inequality discussed in [B1] is a consequence of the so-called 'regular separation inequality', which was first proved by Łojasiewicz for semianalytic sets, and by Hironaka for subanalytic sets. A parameter version of the regular separation inequality was given by Łojasiewicz and Wachta for subanalytic bounded sets. Denkowski's paper [B3] contains a proof of the complex analytic version of Łojasiewicz-Wachta's result using only tools from complex analytic geometry (Theorem 2.1). As an application, in Theorem 3.3, Denkowski gives a parameter version of the Łojasiewicz inequality for c-holomorphic mappings.

In the paper [B5], Denkowski investigates two kinds of 'singular' points of Cartan's weakly holomorphic functions (i.e., functions $f: A \to \mathbb{C}^n$ which are holomorphic on the regular part of the analytic set $A \subseteq \mathbb{C}^m$ and bounded near the singularities). The first kind consists of those points at which a weakly holomorphic function is not 'holomorphic' (i.e., cannot be locally extended to a holomorphic function on the ambient space). The second kind consists of points at which a weakly holomorphic function is not continuous (i.e., points causing the failure of c-holomorphicity). It is proved that the first set of points is analytic (Theorem 3.4), while the second one is only analytically constructible (Theorem 3.1). A new criterion for a weakly holomorphic function to be 'holomorphic' is also given (Theorem 3.3).

Returning to real geometry, in [C4], Denkowski discusses a new condition (weaker than the usual transversality condition) which guarantees, in several special cases, the smoothness of the intersection of two 'subanalytic leaves' (i.e., submanifolds that are subanalytic subsets of $\mathbb{R}^n$).

The paper [C1] takes us back to the Kuratowski convergence in the setting of subanalytic/o-minimal geometry. The main result (Theorem 3.1) gives a necessary and sufficient condition for the upper-semicontinuity of the sections of a closed subanalytic/definable set $E \subseteq \mathbb{R}^k_t \times \mathbb{R}^n_x$. Precisely, it is shown that $\limsup_{t\to 0} E_t = E_0$ iff $\dim_{(0,x)} E > \dim_x E_0$ for any $x \in E_0$.

In Theorem 3.1 of [C5], Denkowski gives a new elementary proof of a theorem of Fremlin about the connectedness of the medial axis of a bounded domain Ω of $\mathbb{R}^n$ (i.e., of the set of points in Ω with more than one closest point to the boundary $\partial\Omega$).

In conclusion, the collection of articles presented by Dr. Denkowski — for the Habilitation procedure [A1–A6] and as 'Other Scientific Achievements' [B1–B5, C1–C5, D1, P1–P7] — represents an important and original contribution to various topics in complex analytic geometry, real geometry (subanalytic sets and o-minimal structures), and singularity theory. In my opinion, the material presented satisfies all the conditions required for Dr. Denkowski to receive the Habilitation of the Jagiellonian University.

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