REPORT ON GREGORZ KAPUSTKA'S MATHEMATICAL WORK

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1. Hyperkähler manifolds

We will review the papers [K2], [K3], [K4] and [K5]. These papers are about hyperkähler manifolds, in particular hyperkähler projective varieties. Together with the group of papers on Calabi-Yau threefolds, this is the most consistent series of papers in Kapustka's bibliography, and it contains some of the best work.

1.1. **Background.** Hyperkähler manifolds (in the present context) are simply connected compact Kähler manifolds carrying a holomorphic *symplectic* form spanning the space of holomorphic 2-forms. Together with compact complex tori and Calabi-Yau varieties, they are basic constituents of compact Kähler manifolds with trivial first Chern class (Beauville-Bogomolov decomposition Theorem).

Two dimensional hyperkähler manifolds were known classically, in fact they are the well-known $K3$ surfaces.

Higher dimensional hyperkählers behave very much like $K3$'s, but much less is known about them. In particular, very little is known about the deformation classes of higher dimensional hyperkählers (Kodaira proved that there is a single deformation class of hyperkähler surfaces). In every (even) dimension greater than two we know two distinct deformation classes (that containing $Hilb^n(S)$ where S is a $K3$, and that containing generalized Kummers), with one extra known deformation class in dimensions six and ten.

As far as we know, there might exist zillions of distinct deformation classes in each dimension higher than two - actually we do not know whether the set of deformation classes in a given dimension (greater than two) is finite.

A problem which is analogous to that of describing the deformation classes of hyperkähler manifolds is the following: give explicit descriptions of locally complete families of *projective* hyperkähler manifolds. This problem is somewhat open also for $K3$ surfaces. Generic low degree polarized $K3$ surfaces are complete intersections (quartic surfaces, complete interescctions of a quadric and a cubic in $\mathbb{P}^4$, etc) or they are generalized complete intersections in homogneous varieties (Mukai). On the other hand, the generic large degree polazrized $K3$'s cannot be parametrized by a rational (or unirational) family, because the corresponding moduli space is of general type (Theorem of Gritsenko, Hulek, and Sankaran).

Such explicit descriptions for higher dimensional hyperkähler manifolds are harder to find, and in fact up to 10 years ago only one such family was known (Beauville-Donagi: lines on a smooth cubic fourfold).

1.2. **Kapustka's contributions.** Kapustka's interest in hyperkähler manifolds was sparked by a project that I initiated in 2008. The goal of my project is to prove the following conjecture: a hyperkähler fourfold X such that $H^2(X;\mathbb{Z})$ equipped with the Beauville-Bogomolov-Fujiki quadratic form is isometric to $H^2(Hilb^2(K3),\mathbb{Z})$, and the Fujiki constant of X is equal to that of $Hilb^2(K3)$ (a *numerical* $Hilb^2(K3)$) is actually a deformation of $Hilb^2(K3)$.

Kapustka produced two papers which make progress towards proving my conjecture, one is [K2], and the other is the preprint "On irreducible symplectic 4-folds numerically equivalent to $Hilb^2(K3)$", arXiv:1004.3177[math.AG].

In order to outline the contents of those papers, I must recall the following results of mine. First I proved that a numerical $Hilb^2(K3)$ hyperkähler fourfold Z can be deformed to a hyperkähler X equipped with a rational (non degenerate) map $f\colon X \dashrightarrow \mathbb{P}^5$, such that one of the following holds:

(1) f is regular, the image Y is an EPW sextic hypersurface, and $X \to Y$ is the natural double cover.

Date: July 4 2017.

(2) f is birational onto a hypersurface Y of degree between 6 and 12, and moreover f is the normalization of Y in the extremal case $\deg Y = 12$.

Secondly, I proved that the natural double cover of a generic EPW sextic is a hyperkähler fourfold, deformation equivalent to $Hilb^2(K3)$. Thus, in order to prove my conjecture, it suffices to prove that Item (2) does not hold.

In his arXiv paper, Kapustka shows that in fact $9 \leq \deg Y \leq 12$, and that the base locus of f is a 0 dimensional subscheme of X, of length $12 - \deg Y$.

The paper [K2] deals with the case $\deg Y = 12$ (which, a priori, is the generic hypothetical case). Kapustka proves an intriguing result, namely that the (unique) adjoint hypersurface to the degree 12 hypersurface Y is an EPW sextic. This is the first and unique significant step forward towards the proof of my conjecture since the appearance of my papers on the subject (almost ten years ago) and Kapustka's arXiv preprint.

The paper [K4] describes a particular EPW sextic and its natural double cover. One of the outcomes of that work is the answer to a question that I had asked regarding maximal pairwise incident planes. The paper has a classical flavour, reminiscent of works on Kummer surfaces, say.

The paper [K5] gives an explicit description of a locally complete family of *projective* hyperkähler deformations of $Hilb^3(K3)$, and is perhaps the most significant contribution of Kapustka up to now. The construction is an interesting variation on the construction of the natural double cover of an EPW sextic.

The paper [K5] suggests that there is a very rich world of hyperkähler varieties associated to Gushel-Mukai fourfolds, in analogy with what is emerging regarding the relation between cubic fourfolds and hyperkähler varieties. More precisely, one associates to a smooth cubic fourfold X the variety of lines on the cubic (the first explicit construction of a locally complete family of projective higher dimensional hyperkähler varieties - Beauville and Donagi), the variety parametrizing twisted cubic curves on X (which has as base Y of a MRC fibration a hyperkähler variety deformation equivalent to $Hilb^4(K3)$, and as X varies the variety Y varies in a locally complete family of projective hyperkählers - C. Lehn, M. Lehn, Sorger, van Straten), a suitably compacified family of intermediate jacobians of smooth hyperplane sections of X (a hyperkähler variety belonging to the exotic 10 dimensional deformation class - Laza, Saccà, Voisin). On the other hand, Iliev and Manivel proved that double EPW sextics are the base of a MRC fibration of the variety parametrizing conics on a Gushel Mukai fourfold, and EPW cubes should be (this is suggested in the paper by Iliev, the Kapustkas and Ranestad) the base of a MRC fibration of the variety parametrizing twisted cubics on a Gushel Mukai fourfold.

2. Other

The remaining work of Kapustka is mainly dedicated to Calabi Yau threefolds. More precisely, there is a long series of papers on explicit constructions of Calabi Yau threefolds. In particular Kapustka produced new examples of Calabi Yau varieties with Picard number one. There are also related results: the construction of an interesting family of surfaces of general type in $\mathbb{P}^5$, explicit constructions of log Fano surfaces, study of certain Fano threefolds. There is a lot of material here, but it is less easily summarized.

3. Final considerations

As explained above, Kapustka has mainly studied hyperkähler and Calabi Yau varieties.

In the field of hyperkähler varieties, one main contribution (perhaps his main contribution) to date is the work on EPW cubes, the other one is given by his two papers on the equivalence of being a numerical $Hilb^2(K3)$ and being a deformation of $Hilb^2(K3)$.

In the field of Calabi Yau varieties, he produced many new examples of Calabi Yau varieties.

Kapustka makes heavy use of algebraic methods developed to study projective varieties. I would classify his mathematics as "applied projective geometry".

Certainly Kapustka has proved to be technically extremely adept. He has pursued with determination two different lines of research. In particular, he entered the field of hyperkähler geometry and quickly became an expert in producing explicit constructions of hyperkähler varieties. His contribution in this field is very original and important.

For all of the reasons explained above, I think that Kapustka has done more than enough to deserve the Habilitation.

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