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REPORT ON THE PHD THESIS THE BERGMAN KERNEL FUNCTION AND RELATED TOPICS BY MARIA TRYBUŁA

The reviewed thesis by Maria Trybuła consists of two parts. In the first part the author occupies with different aspects concerning the theory of Bergman spaces in the domains in $\mathbb{C}^n$. If D is a domain in $\mathbb{C}^n$, the Bergman space in D is thee space of holomorphic functions in D and integrable with the square of modulus with respect to the Lebesgue measure in D. More generally, for a positive function $\alpha : D \longrightarrow (0, +\infty)$,

$$\mathbb{A}^2_\alpha(D) =: \{f \in \mathcal{O}(D) : \int_D |f|^2 \alpha dV < +\infty\},$$

where dV is the Lebesgue measure in $\mathbb{C}^n$.

In Chapter 1.1 the main result, obtained by the author and published in paper [1], is the explicit formula in the symmetrized bidisc $\mathbb{G}_2$. If $s_2 \in [0,1)$ and $X = (X_1, X_2) \in \mathbb{C}^2$, then the Bergman metric $\beta_{G^2}((0, s_2), (X_1, X_2))$ has the form

$$\beta_{G^2}((0, s_2), (X_1, X_2)) = \sqrt{B_1|X_1|^2 + B_2|X_2|^2}, \tag{1}$$

where B_1 and B_2 are some explicitly given rational functions from s_2^2. This is the content of Theorem 1.1.4. The proof of this formula is rather complicated. The main tool are the so called Jacobi - Trudy identities. As the author says, the proof of the analogue formula for (1) for $\mathbb{G}_n$, $n > 2$, should require the overcome of yet more hard numerical difficulties.

Chapter 1.2 is devoted to the investigations of relations between the spaces $\mathbb{A}^2_\alpha(G)$ and $\mathbb{A}^2_{\alpha o \pi}(D)$, where G, D are the domains in $\mathbb{C}^n$, and $\pi : D \longrightarrow G$ is a proper mapping of multiplicity m. In that situation the mapping π induces the operator

$$\Gamma : \mathbb{A}^2_\alpha(G) \longrightarrow \mathbb{A}^2_{\alpha o \pi}(D),$$

given by the formula

$$\Gamma f = \frac{1}{\sqrt{m}}(f \circ \pi) J_\pi, \quad f \in \mathbb{A}^2_\alpha(G)$$

(J_π denotes here the complex jacobian of the mapping π). The main result of Chapter 1.2 is Theorem 1.2.1 (published in the paper [2]), which states that the set $\Gamma \mathbb{A}^2_\alpha(G)$ is a closed subspace of the space $\mathbb{A}^2_{\alpha o \pi}(D)$, which is isometrically isomorphic to the space $\mathbb{A}^2_\alpha(G)$ by the mapping Γ; in the same theorem the formula is given for the orthogonal projection onto the space $\Gamma\mathbb{A}^2_\alpha(G)$.

Theorem 1.2.1 gives as the corollary the formula on the transformation of the Bergman kernels of domains D and G by the holomorphic proper mapping $\pi : D \longrightarrow G$; such a formula was earlier obtained on the another way by S.Bell.

In Chapter 1.3 the author considers the properties of the domain $\mathbb{E}$ in $\mathbb{C}^3$, called the tetrablock. This is the image of the classical domain $\mathcal{R}_{II} = \{\tilde{z} \in M(2\times 2, \mathbb{C}) : \tilde{z} = \tilde{z}^t, \|\tilde{z}\| < 1\}$, by the mapping

$$\varphi : \mathcal{R}_{II} \ni (z_{11}, z_{22}, z) \longrightarrow (z_{11}, z_{22}, z_{11}z_{22} - z^2) \in \mathbb{C}^3.$$

As an application of Bell's transformation rule the author obtaines the result (Corollary 1.3.2) which states the relation between the Bergman kernels of the domain $\mathcal{R}_{II}$ i the tetrablock $\mathbb{E}$. As the corollary it is proved by the author that $\mathbb{E}$ is not a Lu Qi-Keng domain (i.e. the Bergman function $K_{\mathbb{E}}$ of the tetrablock $\mathbb{E}$ has zeroes); this is the content of the Corollary 1.3.3.

The main result of Chapter 1.4 is Theorem 1.4.1; here one obtains some general conditions on the holomorphic mapping F to be proper, and to assure that the image of such mapping to be the open set. The mapping F is defined on the basis of the holomorphic mapping $f : D \longrightarrow \mathbb{C}^n$, where D is a domain in $\mathbb{C}^n$, and f is invariant with respect to some finite group of biholomorphic mappings $\mathcal{U}$; the construction of F uses also some continuous functions $\varphi_j : D \longrightarrow (0, +\infty)$, $j = 1, \ldots, k$, $k \in \mathbb{N}$. As a corollary to this general result the author obtains new proof of the fact that the sets $\mathbb{G}_n$ (symmetrized n-polydisc) and $\mathbb{E}$ (tetrablock) are open sets.

Chapter 1.5 is devoted to the investigations of the properties of the Bergman metric in plane domains in $\mathbb{C}$. The main result of this chapter, Theorem 1.5.8 (published in the paper [3]), concerns the bounded domains $D \subset \mathbb{C}$ with the boundary which is Dini-smooth. The domain with Dini-smooth boundary is the bounded domain $D \subset \mathbb{C}$ such that $\partial D = \gamma^*$, where γ^* is a closed curve which admits the parametrization $\gamma : [-\pi, \pi] \longrightarrow \mathbb{C}$ of class C^1 such that $\gamma' \neq 0$, γ' is uniformly continuous, and γ' is Dini-continuous, i.e. for some $\delta > 0$,

$$\int_0^\delta \frac{\omega(t)}{t} dt < +\infty;$$

here $\omega(t) =: \sup\{|\gamma'(t_1) - \gamma'(t_2)| : |t_1 - t_2| \leq t\}$, $t > 0$ is the modulus of continuity of the function γ'. It is well-known that if a planar domain G has Dini-smooth boundary, then a biholomorphic mapping $F : \mathbb{D} \longrightarrow G$ from the unit disc $\mathbb{D}$ onto G has the derivative F' which extends continuously onto the whole closure of the unit disc $\mathbb{D}$, and $F'(z) \neq 0$ for $z \in \partial\mathbb{D}$. The content of the aforementioned Theorem 1.5.8 is the following estimate for the Bergman metric in domains with Dini-smooth boundary:

If D is a domain in $\mathbb{C}$ with Dini-smooth boundary, then for arbitrary points $z, w \in D$ the following estimates hold:

$$
\begin{aligned}
\text{(2)} \qquad \sqrt{2}\log\left(1+\frac{|z-w|}{c\sqrt{d_D(z)d_D(w)}}\right) &\le b_D(z,w) \\
&\le \sqrt{2}\log\left(1+\frac{c|z-w|}{c\sqrt{d_D(z)d_D(w)}}\right)
\end{aligned}
$$

for a convenient constant $c > 1$; here $b_B(z, w)$ is the distance of points z and w with respect to the Bergman metric in D. The result in Theorem 1.5.8 is the extension of the similar estimate for the Bergman metric for the unit disc in $\mathbb{C}$ (this estimate is the content of Lemma 1.5.4).

The proof of the estimate in Theorem 1.5.8 is highly non-trivial; the author uses the results by Nikolov from the paper [4], the localization principle from [5], Koebe's theorem, and the estimates for some type of metrics in strictly pseudoconvex domains from [6].

The next result presented in Chapter 1.5 (Theorem 1.5.18, published in [3]), concerns the behavior of the Kobayashi k_D, Carathéodory c_D, and Bergman b_D metrics in planar domains: If D is a finitely connected domain in $\mathbb{C}$ and ∂D has no isolated points, then

$$
\text{(3)} \qquad \lim_{w\to\partial D, z\neq w}\frac{b_D(z,w)}{c_D(z,w)} = \lim_{w\to\partial D, z\neq w}\frac{b_D(z,w)}{k_D(z,w)} = \sqrt{2}
$$

uniformly with respect to $z \in D$.

The second part of the thesis is devoted to the investigations of the properties of Kobayashi metric in domains in $\mathbb{C}^n$. In Chapter 2.1 the author recalls the notions of some classes of domains in $\mathbb{C}^n$: $\mathbb{C}$-convex, linearly convex, and weakly linearly convex domains. The author recalls also the notion of the minimal basis for the domain D at the point $q \in D$ and the increasing sequence of n numbers

$$
\text{(4)} \qquad \tau(q) = (\tau_1(q) \le \tau_2(q) \le \dots \tau_n(q)),
$$

characterizing the domain D and the point $q \in D$. The main result of this chapter is Theorem 2.1.3 (published in the paper [7]): If the domain $D \subset \mathbb{C}^n$ does not contain any complex line, q is a point of D, the canonical basis $\mathbb{C}^n$ is a minimal basis for D at q, we have given $r > 0$ and the numbers $\tau_j(q)$, $j = 1, \dots, n$ from (4), then under the assumption that D is weakly linearly convex, we have the following implication:

$$
\begin{aligned}
\max_{1\le j\le n}\frac{|z_j-q_j|}{\tau_j(q)} < \frac{e^{2r}-1}{n(e^{2r}+1)} &\Rightarrow \sum_{j=1}^{n}\frac{|z_j-q_j|}{\tau_j(q)} < \frac{e^{2r}-1}{e^{2r}+1} \\
&\Rightarrow z \in D, \quad l_D(q,z) < r.
\end{aligned}
$$

If D is convex (respectively $\mathbb{C}$-convex), and $z \in D$, then the following implication holds

$$c_D(q, z) \Rightarrow \max_{1 \le j \le n} \frac{|z_j - q_j|}{\tau_j(q)} < e^{2r} - 1$$

(respectively

$$c_D(q, z) \Rightarrow \max_{1 \le j \le n} \frac{|z_j - q_j|}{\tau_j(q)} < e^{4r} - 1).$$

Theorem 2.1.3 gives informations about the reciprocal positions of the balls with respect to the Kobayashi metric.

Chapter 2.2 is devoted to the notion of Gromov hyperbolicity. We say that the distance $d : D \times D \longrightarrow \mathbb{R}_{\ge 0}$ is Gromov hyperbolic if

$$\sup_{x,y,z \in D} (d(x, y) - \min\{d(x, z), \quad d(z, y)\}) < \infty.$$

The author concentrates on the notion of hyperbolicity of the Kobayashi distance in $\mathbb{C}^2$. The main result of this chapter: Theorems 2.2.14 and 2.2.17, were published in the paper [8]. They say that the convex domains with boundary of class $\mathbb{C}^{1,1}$, which have a non-trivial analytic disc in the boundary, or the point of infinite type, with convenient regularity of the defining function, do not satisfy the aforementioned condition of hyperbolicity. In this chapter the author yields yet more examples of non-hyperbolic domains (among them also $\mathbb{G}_n$ - Proposition 2.2.11). Especially important is Proposition 2.2.19, which gives the wide class of hyperbolic, but non-pseudoconvex domains.

The last part of the thesis is Chapter 2.3; here the author gives another fact, which makes worthy to investigate the properties os symmetrized bidisc. Lemma 2.3.1 and following it Remark 2.3.2 show, that the Kobayashi metric for $\mathbb{G}_2$ is not differentiable.

I state that the PhD thesis presented by Maria Trybuła contains a great account of very valuable results from different parts of the theory of Bergman spaces and the theory of holomorphically invariant metrics in domains in $\mathbb{C}^n$. The results are non-trivial, and were obtained by means of another advanced results, obtained by mathematicians which occupy with the considered theory. The author demonstrates in her PhD thesis a very large mathematical skilfulness and a deep knowledge of the theory of invariant metrics, which is now in the region of investigations of excellent specialists in complex analysis.

I state that the PhD thesis by Maria Trybuła satisfies the conditions stated by the law for PhD theses.

At the same time I consider the reviewed PhD thesis by Ms. Maria Trybuła exceptional. Therefore, I kindly request it to be awarded a distinction by the Faculty of Mathematics and Computer Science of the Jagiellonian University.

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