Denis Perrot
Maître de Conférences HDR
Institut Camille Jordan
Université Lyon 1

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Report on the work of Michal Eckstein presented for his PhD Thesis "Spectral Action - beyond the almost commutative geometry"

Mathematical background

A large part of research in non-commutative geometry is devoted to the study of far-reaching generalisations of Riemannian manifolds, called *spectral triples*. The latter are given by an algebra of bounded operators on a Hilbert space, together with an unbounded self-adjoint operator (the "Dirac operator") satisfying some compatibility conditions. A famous reconstruction theorem by A. Connes states that under technical assumptions, spectral triples based on commutative algebras exactly correspond to Riemannian manifolds. On the other hand, spectral triples based on non-commutative algebras allow to extend the tools of ordinary geometry to very exotic objects, including singular manifolds, fractals, quantum spaces, etc...
The aim of this thesis is a thorough study of the analytic properties of zeta functions and heat traces associated to the Dirac operator of very general spectral triples.

Results

The first chapter is a self-contained introduction to the non-commutative geometry "à la Connes". The notion of spectral triple is presented after some prerequisites on operator theory in Hilbert space, which may be very useful to the reader. Then, following the general theory developed by Connes and Moscovici for *regular* (i.e. smooth) spectral triples, the non-commutative differential calculus, abstract pseudodifferential operators, dimension spectrum and non-commutative integral are introduced. This chapter ends with a description of the Chamseddine-Connes spectral action principle, allowing to build realistic physical models of particle physics from almost-commutative spectral triples.
Chapter 2 is based on a joint work of the author with A. Zajac. It contains several important new results about the existence of asymptotic expansions, at small times, for the heat trace associated to abstract Dirac operators. The main result, Theorem 2.2.5, gives sufficient conditions on the zeta-function of the Dirac operator ensuring the existence of an asymptotic expansion of the heat trace. These conditions are often verified in practice. Moreover, the

coefficients of the expansion are given by explicit formulas. A second very important result is Corollary 2.2.6. giving sufficient conditions for the asymptotic expansion to be actually convergent. To my knowledge, the question of existence and convergence of expansions is answered for the first time at this level of generality. The chapter ends with examples of operators illustrating these properties.

Chapter 3 deals with the expansion. at high energies. of the spectral action for regular spectral triples. In Theorem 3.2.1. Eckstein explicitly computes the coefficients of this expansion in terms of higher residues of Wodzicki-Guillemin type. The main improvement compared to the previously known results is that almost no restriction is imposed on the structure of the poles of the zeta-function associated to the Dirac operator. This combined with the results of chapter 2 gives conditions under which the spectral action becomes a convergent series at some energy scales. The latter may have useful applications in the theory of model-building for particle physics.

In chapter 4. Eckstein introduces an abstract pseudodifferential calculus for *quasi-regular* spectral triples, that is, spectral triples with a weakened notion of regularity. While the associated zeta-functions are not so nicely-behaved as in the regular case. most of the results presented in the preceding chapters still apply. The relevance of these notions is illustrated by the very nice example of the Podleś sphere in chapter 5. based on a joint work with B. Iochum and A. Sitarz. This spectral triple is indeed quasi-regular but not regular. and its zeta-function has a very complicated pole structure. Nevertheless, the impressive computation of the heat trace and the spectral action is carried out in detail. and shown to lead to convergent series.

Conclusion

Michal Eckstein demonstrates a great technical skill and a really convincing computational power. combined with ease in the handling of sophisticatd abstract concepts. He obtains new strong results concerning the behaviour of zeta-functions and heat traces associated to spectral triples using a unified approach. Particularly interesting are the completely new results about the convergence of the asymptotic series. with potential application to the building of non-commutative quantum field theories using the spectral action principle. This work presents a clear perpective for future research and I am looking forward to see this program evolve. As a conclusion. this thesis constitutes a great piece of work that easily meets all the requirements for a PhD. Therefore. I strongly recommend to deliver to Michal Eckstein the title of Doctor in Science. with the highest possible distinction.

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