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Review of PhD Thesis

I am writing to provide a review of the PhD thesis submitted by Michał Seweryn. The title of his thesis is *Dimension of posets with cover graphs in minor-closed classes.* As will be clear from the details of my report, Mr. Seweryn's thesis is certainly deserving of "distinction." As a basis of comparison, over my career, I have supervised 14 PhD students and served on the thesis committee of more than 50 other PhD students. Also, I have served as an external evaluator more than 10 times. Mr. Seweryn's thesis ranks among the *top* 5 I have seen over the years.

An Overview of Contents.

In the last 10 years, there has been considerable interest in connections between the combinatorics of posets and structural graph theoretic properties of their cover graphs. Although the roots for this line of research can be traced back to 1970's, modern insights and proof techniques have revealed a deep and fascinating body of knowledge at the boundary of mathematics and computer science. On the poset side, most of the attention has been focused on the Dushnik-Miller concept of dimension, although the concepts of Boolean dimension (and other variations) have also been studied. On the graph side, the concept of graph minors in the Robertson-Seymour sense has been central, although tree-width, planarity, genus and other graph parameters have been studied.

Walczak showed that if $\mathcal{C}$ is a minor closed class of graphs and h is a positive integer, then there is an integer $d(\mathcal{C}, h)$ such that if P is a poset of height h and the cover graph of P excludes graphs from $\mathcal{C}$ as minors, then the dimension of P is at most $d(\mathcal{C}, h)$. An alternative, and more adaptable proof, of this same result was given by Micek and Wiechert. Previously, special cases of this result were known, including the class of planar cover graphs (Streib and Trotter) and the class of graphs with bounded tree-width (Joret, Micek, Milans, Trotter, Walczak, and Wang).

This thesis begins with two preliminary chapters. The first shows clearly that Mr. Seweryn has full mastery of the existing body of knowledge linking dimension of posets and structural graph theory. It also explains clearly his motivation for the particular problems he investigates, and it details their connections with other areas of combinatorial mathematics and theoretical computer science.

The second chapter gives a concise treatment of preliminary—and essential—background material. This chapter is very well done. As a consequence, Mr. Seweryn's thesis is likely to become an important resource for future students and researchers needing a solid and compact discussion of key results and techniques.

Each of the next four chapter includes a comprehensive treatment of a substantive result obtained either independently by Mr. Seweryn or in close collaboration with research colleagues. Where collaboration is involved, it is clear that Mr. Seweryn played a key role in the project. As some of his collaborators are well known senior leaders, this is a important indicator of Mr. Seweryn's stature in the field.

My report will place particular emphasis on the first and last of the four major results—reflecting primarily my interests and expertise. The work on the second and third topic is first rate, but more removed from work that I know well.

Treewidth 2. Trotter and Moore proved that the dimension of a poset whose cover graph has tree-width 1 is at most 3. Subsequently, Kelly showed that for every $t \geq 1$, there is a poset K_t such that the order diagram of K_t is planar, the dimension of K_t is t, the height of K_t is t, and the path-width of the cover graph of K_t is (at most) 3. These results leave open the case of posets whose cover graphs have path-width or tree-width 2. In these cases, is dimension bounded or does it grow with height? Not surprisingly, the path-width case is easier. However, it is bounded in both cases.

Wiechert showed that if P is a poset and the path-width of the cover graph of P is 2, then the dimension of P is at most 6. Joret, Micek, Wang, Wiechert and Trotter proved that if P is a poset whose cover graph has tree-width 2, then the dimension of P is at most 1276. Concerning this paper, I once wrote to a colleague: "This is one of my least favorite papers, as the 30 page proof is complex and unenlightening, some might even say ugly."

Then Mr. Seweryn came to our rescue. First, he lowered the upper bound to 12, evidently a dramatic improvement! Second, and more importantly, Mr. Seweryn's argument is elegant and goes straight to the heart of the problem. This is a paper to be read and studied! The version given in the thesis is polished and very well presented. Also, this work was done by Mr. Seweryn working alone.

Excluding a $K_{2,n}$-Minor. Joret, Micek and Wiechert proved that $d(\mathcal{C}, h)$ grows exponentially in h when $\mathcal{C}$ is the class of graphs that exclude the complete bipartite graph $K_{3,3}$ as a minor. This leaves open the possibility that $d(\mathcal{C}, h)$ might be independent of h when $\mathcal{C}$ is the class of graphs that exclude the complete bipartite graph $K_{2,n}$ as a minor, for some fixed $n \geq 2$.

Mr. Seweryn verifies this independence from height using a variant of a result of G. Ding which characterizes the graphs in this family. Mr. Seweryn's proof is elegant and appealing. Also, his adaptation of the preliminary work of Ding is non-trivial and has independent merit.

Excluding a Ladder.

A $2 \times n$-grid is called a ladder. Mr. Seweryn proves (in joint work with Huynh, Joret, Micek and Wollan) that $d(\mathcal{C}, h)$ depends only on n when $\mathcal{C}$ is the class of posets whose cover graph exclude an n-ladder. The proof uses connections with concepts called tree-depth and centered colorings. Although similar in spirit with the previous work on $K_{2,n}$, the result for ladders is completely different in the details.

Outer-planarity.

While this reviewer feels the entire thesis is first rate, the work on this topic shows, in my opinion, special creativity and insights. It centers on the class of posets with planar cover graphs, i.e., the class $\mathcal{C}$ consists of graphs not containing either K_5 or $K_{3,3}$ as minors. As noted previously, Streib and Trotter proved that the function $d(\mathcal{C}, h)$ exists—although the original proof yielded only a tower function of h as an upper bound.

Subsequently, the work of Joret, Micek, Milans, Trotter, Walczak and Wang on tree-width yielded a bound which is doubly exponential in h^2. In turn, the foundational paper of Micek and Wiechert for the class $\mathcal{C}$ of graphs excluding the complete graph K_n as a minor lowered the bound to double exponential in h. In turn, the work of Joret, Micek and Wiechert on sparsity and dimension lowered the bound to a single exponential of the form $2^{\mathcal{O}(h^3)}$.

The first polynomial bound was obtained by Kozik, Micek and Trotter. Their bound is $\mathcal{O}(h^6)$, and their proof contains a key lemma showing that the dimension of a doubly exposed poset is bounded in terms of the standard example number *independent* of height. It also proves that this bound is at most quadratic. However, the approach taken in this paper also includes a dependency on weak-coloring methods, and this comes at a cost of a multiplicative error which may be as large as $\mathcal{O}(h^3)$.

Mr. Seweryn (in joint work with M. Gorsky) found a much better way. Given a drawing in the plane without edge crossings of a planar graph G, there is a natural definition of the graph being *k-outerplanar*. This means that it takes k iterations to hit all vertices, if at each step, we remove the vertices that are on the outer face. In particular, an outerplanar graph is 1-outerplanar. Seweryn then proved that if P is a poset and the cover graph of P is k-outerplanar, then $\dim(P) = O(k^3)$. The result for height follows since it is easy to see that if P is a poset with a planar cover graph and the height of P is h, then P is k-outerplanar for some $k \leq 2h$.

This work also involves a notion of an "internal folding", one that we are happy regardless of whether a heavy pair of cells in an unfolding tilts left or tilts right. This concept is very likely to have other applications. More generally, the notion of k-outerplanarity is a special case of a *layering*, and this construct is widely used in structural graph theory.

Seweryn's proof actually shows that for the class $\mathcal{C}$ of posets with planar cover graphs, $d(\mathcal{C}, h) = \mathcal{O}(hf(h))$, where $f(h)$ is the maximum dimension among doubly exposed posets with standard example number h. The bound of Kozik, Micek and Trotter on $f(h)$ is $\mathcal{O}(h^2)$, leading to the upper bound $\mathcal{O}(h^3)$. As noted in his thesis, Smith Blake and Trotter have recently announced an improvement to $f(h) = \mathcal{O}(h)$. As a consequence, the Seweryn approach now shows that the dimension of a poset with a planar cover graph is bounded by a quadratic function of its height. From below, we have always had a linear lower bound. While there is still work to be done, this body of work, capped by the Seweryn approach, is a formidable body of work constructed by multiple researchers over 15 year time span.

Summary Recommendation This is an *excellent* PhD thesis and is *certainly* deserving of being designated as “distinguished.” Mr. Seweryn already has an impressive body of accomplishments including the four beautiful results featured in this thesis. Some of his results have been obtained in direct competition with researchers who are much more senior in age and experience. Others have been obtained in close collaboration with colleagues. Some of them are peers (Gorsky, e.g.,) and others are established senior researchers (Joret, Micek and Wollan, e.g.). Both his potential and his promise are enormous.

Please feel free to contact me by email/skype/zoom, etc., if I can clarify any aspect of this review.

Sincerely,

William T. Trotter

William T. Trotter
Professor Emeritus