KTH Engineering Sciences

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Report on doctor thesis of Mateusz Przybylski:
Metody teori indexu Conleya w dynamice próbkowej

It is my opinion that the thesis deserves the passing grade.

Overview of the thesis. In reality outcomes of experiments are recorded by necessarily finite number of measurements. To approximate and possibly describe some aspects of dynamics governing a particular experiment one collects finite number of such measurements at finite number of time slots. How to interpret such a discrete approximation and extract information about possible dynamics behind the experiment is therefore a central challenge which is essential in applications. It could be that recovering or identifying a self map generating the dynamics is either out of reach, or very costly or simply not possible. In his thesis, Mateusz Przybylski discusses and uses techniques of how to tackle this challenge based on join work with Batko, Mischaikow, and Mrozek. Part of this work has been already published (reference [3] in the bibliography). There is also a new preprint available which I have not been able to look at. I strongly believe the results presented in the thesis describing the content of [3] are *important*. It is based on a brilliantly simple way of extracting reliable information about the dynamics from its discrete subsampling without the need of identifying directly the self map $X \to X$ generating the dynamics. Instead of such a map, one looks for a zig-zag $X \xleftarrow{f} Y \to X$ where f has acyclic fibers. Although such a zig-zag might not lead to a geometric map $X \to X$, it leads to a self map on the level of homologies $H(X) \xrightarrow{H(f)^{-1}} H(Y) \to H(X)$. This self map on homologies can be then used for example to describe the number of isolating neighbourhoods, attractors, fixed points etc in the spirit of Lefschetz fixed point theorems, Conley index theory and Ważewski property, which is also presented in the thesis. Extracted dynamical information is packaged into a finite graph and section 4 of the thesis recalls and explains how this is done. The simplicity of this idea, in my opinion, will make a lasting contribution in the way dynamical properties are extracted from discrete observations.

To build an appropriate zig-zag from a finite time series of measurements is not however an easy task. There are both mathematical and computational challenges. Both of these challenges are addresses and discussed in the thesis. One way of thinking about the map $f\colon Y \to X$, in a zig-zag, is as a multi valued function $X \multimap Y$ given by the fibers of f. That is why multivalued functions play an important role in the thesis, which contains an extended discussion of how to do topology and dynamics on such functions. This part of the thesis is based on work described in references [1] and [2], and takes advantage of generalised definition of an isolating neighbourhood (Definition 3.10) and appropriately generalised Conley index theory taken from [2].

The key examples of multivalued functions considered in the thesis are cubical functions constructed in two steps: first by using Takens theorem to convert a finite time series of measurements into a subset of the Euclidean space, and then approximating its image by cubical complexes. In Takens

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theorem the assumption about genericity of the input (self diffeomorphism $\phi\colon M \to M$ and a smooth function $f\colon M \to \mathbb{R}$) is crucial. This theorem is not true for ϕ being the identity or for f being a constant function. I therefore find it unacceptable the way this theorem is presented in the thesis (Theorem 2.1) where this assumption is not mentioned at all in the theorem, and only a short remark is written at the end of the following paragraph. That should be adjusted.

Computational and algorithmic aspects of this approach to extract information about dynamics describing a finite time series of measurements is extensively discussed in the thesis. I particularly like Sections 5 and 6. Section 5 because of Theorem 5.25, and section 6 because of beautiful and informative illustrations of the presented theory.

There is also section 7 in the thesis, which has a bit different focus than the rest. It aims at describing dynamics and Conley index theory on a binary relation on a finite set, or equivalently on a directed simple (max one edge between two vertices) graph. It builds on work of Szymczyk in [55]. This section gives foundational results needed for setting up Conley index theory á la Szymczyk. Although technical, such work is in fact needed and I appreciate that this has been done in the thesis. I do hope that it opens a stage for interesting future results of less technical nature.

Conclusion. Sections 4, 5, and 6 of the thesis contain an account of join work of Mateusz Przybylski with Batko, Mischaikow, and Mrozek. I find the mathematical content of this part of the thesis important and of the highest quality. Part of this work has already been published in a very high impact SIAM Journal on Applied Dynamical Systems. There is also an additional preprint available with the same set of authors.

Section 7 is foundational and describes basics of how to set up Conley index theory á la Szymczyk on a simple directed graph. I believe this work is valuable and with a potential of further development.

The thesis is at some places well written and at some places rather technical and not easy to read. Illustrative examples help in absorbing presented math. Stated Takens theorem 2.1 should be adjusted as there are important assumptions missing. The thesis also contains an introductory part describing its context. I do appreciate this part of the thesis. It is well written and structured providing a valuable resource to other students and the community at large, which I believe is an important aspect of writing a doctoral thesis.

Crucial part of the thesis in based on collaborative work and I am unable to judge the contributions of Mateusz Przybylski. I can only base my opinion on the content of the thesis itself. It is my opinion that the quality of presented math in the thesis deserves acceptance of the thesis for the doctor degree in mathematics.

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