REPORT ON PH.D. THESIS "MORSE-CONLEY-FORMAN THEORY FOR GENERALIZED COMBINATORIAL MULTIVECTOR FIELDS ON FINITE TOPOLOGICAL SPACES"

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1. Background and Goals

The field of dynamical systems is concerned with the long-term behavior of systems that evolve over time. This behavior can be extremely complicated, and it is often necessary to use computers to help us understand it. A powerful general method is to approximate the given system with an appropriate combinatorial model, analyze the dynamics of that model, then draw conclusions about the original dynamics from the model dynamics. This approach has been extremely useful for discrete-time dynamical systems (self-maps), for example by discretizing the space, constructing an outer approximation to the map, analyzing the resulting combinatorial dynamical system, and applying the Conley index to get rigorous results about the original map.

For continuous-time dynamical systems (flows generated by vector fields, generally arising from differential equations), this approach has been more challenging. In particular, it is not at all clear what an appropriate combinatorial model of a vector field might be. Existing methods can be too limited to apply to all systems of interest, or too complicated to be useful. This thesis offers a method, combinatorial multivector fields, defined on general finite topological vector spaces, including simplicial complexes. It gives analogues of dynamical notions and results from classical dynamical systems theory.

2. Description

Chapter 1, "Preliminaries," explains the notation and background material on multivalued maps, directed graphs, partially ordered sets, finite topological spaces, and algebra. Most of this is standard, but the material on finite topological spaces is less common in the field of dynamical systems (I at least was not very familiar with it), and I appreciated the brief and understandable discussion.

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Chapter 2, "Algebraic topology," gives the necessary background in homology. Again, much of this was familiar, but the application to order complexes and finite topological spaces was mostly new to me, and I found it very helpful. There is also a very brief discussion of zigzag persistence.

Chapter 3, "Dynamical systems," gives definitions and examples of the objects of study. It starts with a brief review of the topological theory of continuous dynamical systems, including isolated invariant sets, the Conley index, limit sets, attractors and repellers, Morse decompositions, and Morse-Conley graphs. These notions are the model that this thesis generalizes. The second section offers a very minimal discussion of combinatorial dynamical systems, just the definition and the notion of solutions, leaving the more sophisticated dynamical notions for the next chapter. It ends with examples of combinatorial dynamical systems arising from three different combinatorial analogues of vector fields. The first is Forman's original definition for CW-complexes. The second is Mrozek's notion of combinatorial multivector fields on Lefschetz complexes, designed to model a greater range of dynamical systems. The third generalizes these combinatorial multivector fields to arbitrary finite topological spaces, removes a technological restriction (the requirement that each multivector contain a unique maximal element), and simplifies the definition of the induced multivalued map. The examples provide good intuition about these multivector fields, preparing the reader for the details in the next chapter, and about the ways in which this new definition is an improvement over the previous one.

The heart of the thesis is Chapter 4, "Combinatorial multivector fields theory." It defines dynamical notions for combinatorial multivector fields analogous to those for classical dynamical systems and gives proofs of their properties. The first section gives the new definition of combinatorial multivector fields on finite topological spaces and defines the crucial notions of regular and critical multivectors and essential solutions, which lead to the definitions of invariant sets and isolated invariant sets. It explains the correspondence between these notions and directed graphs. The second section gives the definitions and necessary results for index pairs and the Conley index. The crucial result is that with these definitions, the Conley index of an isolated invariant set does not depend on the choice of index pair used to compute it. Section 4.3 deals with attractors, repellers, and α- and ω-limit sets. The next section defines Morse decompositions. Again, the notion of essential solutions is crucial; using more general solutions leads to the related weak Morse decompositions. Properties analogous to those of classical Morse decompositions, including the Morse inequalities, are defined and proven.

Chapter 5, "Persistence of Morse Decomposition," deals with zigzag persistence modules for the homology of Morse decompositions. For technical reasons, it is necessary to use weak Morse decompositions (which may contain too many elements to be useful in practice) or to relax the assumptions on the intermediate spaces in the zigzag diagram. Since combinatorial Morse sets are not necessarily closed or disjoint, the disconnecting topology is necessary for these constructions. This chapter is brief, and I would have found more explanation and examples helpful, but that may just be because of my lack of expertise in zigzag persistence.

Finally, Chapter 6, "Numerical experiments," gives four examples of applications to vector fields sampled from ordinary differential equations. As the author notes, these serve as a proof of concept, rather than rigorous results about the relationship between the classical vector field and the combinatorial multivector field approximation. The first step is to build a multivector field from the set of sampled vectors. A method is presented, based on the greedy algorithm and involving an angular parameter that governs the size of the multivectors and thus the "thickness" of solutions. The first example is a straightforward calculation of the Morse-Conley graph, which shows reasonably good agreement with the actual dynamics of the differential equation. The remaining three examples compute persistence diagrams, using as parameters the amount of random noise added to the system, the angular parameter, and a parameter in the equations for the Sel'kov glycolysis model.

3. Comments

This is a valuable contribution to our ability to understand continuous dynamical systems by approximating them with computationally tractable models. In order to do that, we need a working theory of continuous dynamics for finite topological spaces, and this work offers a strong, well-developed model. Trying to find the "right" definitions is not easy – if it were, it would have been done decades ago. They have to be simple enough to understand, compute, and work with, but strong enough to give meaningful results. It is important to note that although many of the results given are analogous to those for the classical setting, the proofs are not in general just straightforward adaptations of the existing proofs; instead, they show real originality and deep technical knowledge.

The work also covers an impressive breadth of topics, in addition to classical dynamical systems and Conley index theory (which is already both broad and technically challenging), including point-set topology, algebraic topology, the algebra of persistence modules, and numerical and computational techniques. This breadth is part of what makes these results so valuable, as they bring together ideas and methods from a number of different disciplines to help us understand the complex behavior of dynamical systems.

This thesis provides a well-developed analogue of classical topological continuous dynamics for the combinatorial setting. This analogue is interesting for its own sake, but its ultimate usefulness will depend on whether it can be applied to give rigorous results for underlying classical dynamical systems, and that is not addressed in this work. If we construct a multivector field from a sample of a classical vector fields, what, if anything, do the combinatorial dynamics tell us about the classical dynamics (which are what we are really interested in)? For example, does a combinatorial invariant set imply the existence of a classical invariant set in the space modeled by the simplicial complex? Is there a relationship between the combinatorial Conley index and the classical Conley index? Part of the power of the discrete Conley index is that it can be used in this way: an outer approximation of a map can give rigorous information about the dynamics of the actual map. Without that connection, it is not clear what these results will ultimately be useful for. (The examples given seem to show a good correspondence between the actual and combinatorial dynamics, which is promising, but of course does not necessarily hold in general.) This is not really a criticism of this thesis. It absolutely provides a valuable contribution to the long-time program of using computational methods to draw rigorous conclusions about continuous dynamics; the fact that there is still more to do does not lessen the achievement.

The thesis is well written and clear. It does a good job of explaining the ideas and intuition, not just the formal details. The examples are well chosen, and the figures are clear and helpful.

There are some minor grammatical errors, but the meaning is always clear. In section 6.3.3, it is stated that "small values of the angle parameter produce a thin and more precise orbit" – if I understand the algorithm correctly, I think that it should read "large values." Similarly, "increasing the parameter leads to more expansive multivector fields" should read "decreasing the parameter."

4. Conclusion

In my opinion, this work clearly meets the international standard for a Ph.D. degree. It is a novel, substantial, and valuable contribution to the field of topological and computational dynamical systems. I recommend that it be granted a distinction.

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