ADAM MICKIEWICZ UNIVERSITY IN POZNAŃ
Faculty of Mathematics and Computer Science
Departament of Algebra and Number Theory

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Report on the thesis of Bartosz Sobolewski,
Last nonzerodigits and p-adic valuations of special sequences

This thesis primarily concerns investigations of the last nonzero digits in the b-ary expansions of several interesting sequences such as factorials, linear recurrence sequences, polynomials, and p-adic analytic functions evaluated at consecutive integers. A common theme in this thesis is the investigation of automaticity or regularity of these last digit sequences. The author proves many results that are more general than those found in the literature.

This dissertation is broken up into four chapters and based on two published papers [69, 70] and unpublished work of the author. The first chapter introduces automatic and regular sequences and gives the basic properties that are used later. After this, basics of p-adic analysis are summarized. The chapter closes with a survey of linear recurrence sequences. I feel this introduction is well written. Of course, reading the rest of the dissertation will require much more experience with these topics.

This dissertation contains many results with very specific conditions and many examples. I will give a few examples of interesting theorems, but I need to emphasize that this report only includes a small sampling. Chapter 2 focuses on the last nonzero digits of factorials and is based on the paper [70] by the author which was published in Acta Arithmetica. Let ν_b denote the b-adic valuation. Define the function $\ell_{b,d} : \mathbb{Z} \to \{0, 1, \dots, b^d - 1\}$ by

$$\ell_{b,d}(m) = \begin{cases} b^{-\nu_b(m)} m \bmod b^d & \text{if } m \neq 0 \\ 0 & \text{if } m = 0 \end{cases}.$$

We use the convention that ℓ_b denotes $\ell_{b,1}$. Many very interesting known results are listed. One that left my thinking when I was it was due to Deshouillers and Ruzsa [29] where they proved, among other things, that the sequence $((\ell_{12}(n!))$ coincides with a 3-automatic sequence on a set of integers n of density 1. For a positive integer b, consider the prime factorization

$$b = p_1^{l_1} \cdots p_s^{l_s},$$

where $p_1, \dots, p_s$ are distinct prime numbers and if there are at least two primes, we reorder so that

$$l_1(p_1 - 1) \geqslant l_2(p_2 - 1) \geqslant \cdots \geqslant l_s(p_s - 1)$$

and

$$p_1 = \max\{p_i : l_i(p_i - 1) = l_1(p_1 - 1)\}.$$

Let

$$\mathcal{B} = \{b \geqslant 2 : s = 1 \text{ or } l_1(p_1 - 1) > l_2(p_2 - 1)\}.$$

Lipka proved in [29] that $(\ell_b(n!))$ is p_1-automatic if $b \in \mathcal{B}$ and nonautomatic otherwise. Theorem 2.2 of the dissertation improves this by additionally showing that if $b \notin \mathcal{B}$, it is not automatic, but coincides with a p_1-automatic sequence on a set of density 1. There are many other interesting results and examples in this chapter.

The third chapter concerns the last nonzero digits of polynomials and p-adic analytic functions. With more generality, we define for any $x \in \mathbb{Q}_b$

$$\mathcal{L}_b(x) = \begin{cases} b^{-\nu_b(x)}x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

and

$$\ell_{b,d} = \mathcal{L}_b(x) \bmod b^d.$$

The main questions explored are

1. When is $(\ell_{b,d}(f(n)))$ a k-automatic sequence?
2. When is $(\mathcal{L}_b(f(n)))$ a k-regular sequence?

I will give one example of a theorem in this chapter. Generally, they are a bit longer to state, so I won't write them all down. Let

$$\mathcal{A}_b = \{f = (f_1, \dots, f_s) : f_i : \mathcal{Z}_{p_i} \to \mathcal{Q}_{p_i} \text{ is strictly analytic on } \mathbb{Z}_{p_i}, f_i \neq 0 \forall i \in [1, s]\}.$$

Theorem 3.17 states that if $f = (f_1, \dots, f_s) \in \mathcal{A}_b$, then

1. If for some i the function f_i has no root in $\mathbb{Z}_{p_i}$, then for all $d \geqslant 1$ the sequence $(\ell_{b,d}(f(n)))$ is periodic.
2. Assume that there exists $\theta \in \mathbb{Q} \cap \mathbb{Z}_b$ such that for each $i = 1, \dots, s$ the number θ is the only root of f_i in $\mathbb{Z}_{p_i}$ and has multiplicity $m_i \geqslant 1$. Let $w_1, \dots, w_s$ be positive integers satisfying $m_1 w_1 = \ldots = m_s w_s$ and put $k = b_1^{w_1} \cdots b_s^{w_s}$. Then for all $d \geqslant 1$ the sequence $(\ell_{b,d}(f(n)))$ is k-automatic and not l-automatic for any $l \geqslant 2$ multiplicatively independent with k.
3. Otherwise, the sequence $(\ell_{b,d}(f(n)))$ is not k-automatic for any $d \geqslant 1$ and $k \geqslant 2$.

The chapter concludes with many specific examples.

The final chapter is based off of the author's paper [69], published in Journal of Number Theory. In this chapter, the 2-adic valuation of linear recurrence sequences is considered. Let $t_n(k)$ be defined as follows: $t_0(k) = 0, t_1(k) = \cdots = t_{k-1}(k) = 1$ and for any $n \geqslant 0$ by the recurrence

$$t_{n+k}(k) = \sum_{i=0}^{k-1} t_{n+i}(k).$$

For example, it is shown that for $k \geqslant 4$, even that for all $n \geqslant 0$

$$\nu(t_n(k)) = \begin{cases} 0 & \text{if } n \equiv 1, 2, \dots, k \pmod{k+1} \\ 1 & \text{if } n \equiv k+1 \pmod{2(k+1)} \\ \eta_2(n) + \eta_2(k-2) + 1 & \text{if } n \equiv 0 \pmod{2(k+1)} \end{cases}.$$

This chapter concludes with numerous interesting applications that I couldn't hope to even begin listing here.

I wanted to remark that the English and writing are quite good. There are many common mistakes that I could have seen, but didn't see. So, I'm not going to include any list of corrections as I feel that it's already more than good enough from this perspective.

The results proven in this disseration are well motivated and strengthen those already in the literature and are far more numerous than I have listed here. The proofs are interesting and nontrivial. Moreover, the author has published much of this dissertation in good journals (Acta Arithmetica and Journal of Number Theory). My overall opinion about the work of Bartosz Sobolewski is positive, the dissertation under review fulfils all the requirements for a PhD thesis, and I recommend to accept it.

Sincerely Yours,

Signature Not Verified

Dokument podpisany przez William Mance
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prof. UAM dr hab. William Mance