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Report on Artur Piękosz's habilitation thesis

"Generalized topology in the sense of Delfs and Knebusch"

The starting point of the research works constituting this habilitation thesis is the generalization to the o-minimal framework of H. Delfs and M. Knebusch's theory about *locally semialgebraic spaces*, which are the spaces that are locally isomorphic to a semialgebraic subset of R^n, where R is an arbitrary real closed field. Delfs and Knebusch introduced these spaces in the 80's of the previous century and achieved many results of topology and homotopy theory in this category. One can regard their approach as a study of semialgebraic geometry in the non affine case.

O-minimal structures appeared in the 80's (at the same period as locally semialgebraic spaces) and were intensively studied by many authors, who described the topology of the singularities of sets and mappings occurring in these structures. The sets which are definable in these structures can be regarded as generalizations of semialgebraic or subanalytic sets which had been the focus of interest of many singularists in the 70's, especially in Cracow. Furthermore, o-minimal geometry is sometimes considered as a realization of Grothendieck's program about tame topology ("topologie modérée").

The work [H1] introduces the notion of *locally definable spaces* (with respect to an o-minimal structure expanding a real closed field), which is the natural generalization of Delfs and Knebusch's notion of locally semialgebraic spaces to the o-minimal category. The possibility of extending Delfs and Knebusch's results to this framework was actually suggested by Knebusch himself and had been partially undertaken: M. Edmundo had investigated the locally definable groups (in o-minimal structures) while E. Baro and M. Otero had studied locally definable homotopies. Some results about definable homotopy theory were also achieved (see [12,13,14] in the self-report). Piękosz's work however is the first one that offers the full generalization of Delfs and Knebusch's theory. It also extends to the o-minimal category Knebusch's theory about weakly semialgebraic spaces.

Delfs and Knebusch's definition of locally semialgebraic spaces relied on the notion of *generalized topological spaces*. A generalized topological space is a space that satisfies all the properties of a topological space except one: the union of an infinite family of open sets is not necessarily open. Only some specific (infinite) coverings, called admissible, are required to form an open set by gluing together. Additional conditions are then required on admissible coverings.

The detailed approach of [H1], explaining all the basic necessary definitions and giving many examples illustrating the purpose, makes an elegant presentation of the theory. Topology on real closed fields, which can be totally disconnected, is more delicate than topology on $\mathbb{R}$. One can however regret that no effort was made to extend the known theories (locally semialgebraic or locally o-minimal), the proof of the geometric facts being patterned on their semialgebraic counterparts. Some results are mentioned without proof, referring to the locally semialgebraic or o-minimal case for them. The o-minimal generalizations that are carried out do not require to overcome challenging problems by developing specific techniques and the proofs of the geometric statements (about homotopy, topology of locally or weakly definable sets) are analogous to those of the semialgebraic case. The level of generality offered by o-minimal structures nevertheless makes the generalization to this category valuable for applications. Developing them would have brought some originality to the research.

The next two articles of the thesis under review [H2,H3] are devoted to the study of the generalized topological spaces as such. The author underlines with these two works the topological aspect of the theory of locally semialgebraic spaces and weakly semialgebraic spaces. The presented theory unravels the interplay of Delfs and Knebusch's generalized topological spaces (GTS) with the locally definable spaces and sheds light on the relevance of the GTS's.

The article [H2] starts with a discussion about the axioms and two formulations are proposed, together with an outline of Grothendieck topology, introducing adequate systems of notations and providing several examples. Terminology is introduced, basic properties are established. The good acquaintance of the author with the abstract language of categories makes a pleasant reading and a clear presentation.

The author took a lot of care to delimit the study by many examples and counterexamples. The high level of generality offered by this notion of generalized topological spaces however somewhat makes it difficult to exhibit any intrinsic rigidity of the theory, and the main results are statements in the language of categories, establishing for instance that the category GTS is topological (as a construct). The motivation and the relevance of such results were actually not completely obvious to me, as I could not really see whether the ultimate aim consists of applications to the (locally, weakly) o-minimal categories or to the non (locally, weakly) o-minimal categories. The proofs of these articles are actually fairly short. The novelty indeed rather lies in the idea of the study, which emphasizes the role played by the generalized topological spaces in the theory of locally definable spaces.

The last two articles of the habilitation thesis [H4-5], together with the article [P9], constitute the most original part of the research works. Piękosz, in joint works with E. Wajch, set about studying *generalized topology*.

In [H4], the authors investigate the natural question of compactifications of generalized topological spaces. The idea is to use Wallman type compactifications. For instance, given a generalized topological space which is homeomorphic to a product of Tychonoff spaces, they show [H4-Theorem 2.9 (ii)] that the associated Wallman space is compact if and only if the so-called ultrafilter theorem holds (that every filter can be enlarged to an ultrafilter). It is noteworthy that the issue of the axiom of choice (which is not assumed in the latter result) is discussed all along the article. Other compactification results are provided. This kind of approach, if pushed further, might prove relevant: the real spectrum (whose compactness relies on the axiom of choice since it requires Tychonoff's theorem) and other related techniques (often relying on the ultrafilter theorem), are very useful to semialgebraic geometers. No attempt to get more concrete results was however made and one more time the motivation for carrying out this research work is not conspicuous. The generality of the study seems again to hamper the achievement of more concrete results.

In [H5], together with [P9], the authors address the question of the quasi-metrizability of a generalized topological space. Quasi-metrization theorems are provided and the question of the axiom of choice is again discussed, with some examples provided. The authors show again (like in [H3]) that some categories are topological constructs.

This research work extends Delfs and Knebusch's theory to o-minimal structures, completing works which had been partially undertaken. It also revisits the theory of locally semialgebraic spaces (or locally definable spaces) from a purely topological point of view and develops the theory of generalized topological spaces, providing several examples, and establishing new results on this topic. For these reasons, despite the lack of original challenging achievement or significant improvement of the results which are known in the semialgebraic framework, I by and large rate these achievements as acceptable to obtain the *"Doktor Habilitowany"* degree.

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