Referee's report for the Habilitation
of Dr. Anna Valette

The material presented by Dr. Anna Valette as the basis for the Habilitation procedure comes within the theme "Asymptotic sets in real and complex geometry" and is organized around of 7 papers [A1]-[A7]. (The citations in my report are the same as in the "Summary of Professional Accomplishments" provided by A. Valette).

· The first two papers, [A1] and [A2], are related to the Jacobian conjecture which asserts that any polynomial mapping $f: \mathbb{C}^n \to \mathbb{C}^n$ with nonzero constant Jacobian has a polynomial inverse, or equivalently, is proper. Formulated by Keller 80 years ago, this conjecture is still unsolved although many attempts of proof were given. In the early 1980s, Wang proved it for polynomials of degree ≤ 2, and Bass, Connell and Wright showed that if the conjecture is true for polynomials of degree ≤ 3, then it is true in general. Moh showed that the conjecture is satisfied for polynomials in 2 variables of degree ≤ 100.

In [A1], A. & G. Valette proposed an interesting new approach. Using a result of Mostowski ("separation lemma"), they constructed a pseudomanifold N_f associated to a given polynomial mapping $f: \mathbb{C}^n \to \mathbb{C}^n$ (Proposition 2.3), and in the special case where $n = 2$, they proved that a polynomial mapping $f: \mathbb{C}^2 \to \mathbb{C}^2$ with nowhere vanishing Jacobian is not proper if and only if the homology, or the intersection homology, of N_f is nontrivial (Theorem 3.2).

Combined with a previous result of G. Valette, which yields, for any compact subanalytic pseudomanifold X, an isomorphism between the L^∞ cohomology of the regular part of X and the intersection cohomology in the maximal perversity of X itself, the theorem above implies that a polynomial mapping $f: \mathbb{C}^2 \to \mathbb{C}^2$ with nowhere vanishing Jacobian is not proper if and only if the L^∞ cohomology of the regular locus of $N_f \cap \bar{B}_R(0)$ is nontrivial (Corollary 3.4). Here, $\bar{B}_R(0)$ denotes a sufficiently large closed ball.

These results were generalized to the n-dimensional case for an arbitrary n in Theorem 4.5 and Corollary 4.11 of the paper [A2] with Nguyen Thi Bich Thuy.

· In the article [A3], A. & G. Valette presented a new generalized version of the Sard theorem on a real closed field R. The (well known) semialgebraic version of Sard's theorem asserts that the set of critical values of a C^1 semialgebraic mapping $f: R^n \to R^k$ has dimension $< k$. (The critical values of f are the images by f of the elements at which f fails to be a submersion.) Using the function ν of Rabier, which measures the distance of a linear operator to the set of singular operators, A. & G. Valette introduced in [A3] the notion of

"z-critical value" of a C^1 semialgebraic mapping $f: X \to R^k$, where $X \subset R^n$ is a bounded semialgebraic manifold. More precisely, $y \in R^k$ is a z-critical value if it is the image by f of a point $x \in X$ such that $\nu(d_x f) < z$. Here, $z \in R$ is an "infinitesimal", that is, z is positive and smaller than any positive rational number. The main theorem of [A3] (Theorem 3.2) is a Sard type theorem for these critical values. The size of the set of these critical values is estimated using the notion of "v-thin set" introduced in a previous paper by G. Valette. Precisely, the theorem says that if v is a convex subgroup of R, then for any infinitesimal $z \in v$ the set of z-critical values of f is v-thin or of dimension $< k$.

As an application of this result, A. & G. Valette derived a Sard type theorem for the "asymptotic critical values" of a C^1 semialgebraic mapping $f: X \to \mathbb{R}^k$, where X is a (not necessarily bounded) C^1 semialgebraic submanifold of $\mathbb{R}^n$. The set $K(f)$ of "generalized critical values" of f, which is defined by

$$K(f) := \{y \in \mathbb{R}^k \mid \exists (x_i)_{i\in\mathbb{N}} \text{ in } X,\ f(x_i) \to y \text{ and } (1 + |x_i|)\nu(d_{x_i} f) \to 0\},$$

contains not only the set $K_0(f) := \{y \in \mathbb{R}^k \mid \exists x \in f^{-1}(y),\ \nu(d_x f) = 0\}$ of critical values of f but also other elements that are called "asymptotic critical values". In particular, $K(f)$ contains the set $K_\infty(f)$ of asymptotic critical values at infinity, which is defined by

$$K_\infty(f) := \{y \in \mathbb{R}^k \mid \exists (x_i)_{i\in\mathbb{N}} \text{ in } X,\ |x_i| \to \infty,\ f(x_i) \to y \text{ and } |x_i|\nu(d_{x_i} f) \to 0\},$$

and if $K_1(f)$ denotes the set of asymptotic critical values at the points of $\overline{X} \setminus X$, which is defined by

$$K_1(f) := \{y \in \mathbb{R}^k \mid \exists (x_i)_{i\in\mathbb{N}} \text{ in } X,\ |x_i| \to \overline{X} \setminus X,\ f(x_i) \to y \text{ and } \nu(d_{x_i} f) \to 0\},$$

then $K(f) = K_0(f) \cup K_1(f) \cup K_\infty(f)$. Using their version of the Sard theorem for the z-critical values, A. & G. Valette were able to prove that the set of asymptotic critical values of a C^1 semialgebraic mapping $f: X \to \mathbb{R}^k$ as above has dimension $< k$ (Theorem 4.3). A similar result was also obtained by Kurdyka, Orro and Simon who showed that the set of asymptotic critical values at infinity has dimension $< k$.

· The starting point of the paper [A4] is a famous theorem of Hardt which says that any continuous semialgebraic mapping is locally semialgebraic trivial (i.e., admits local semialgebraic trivializations) outside some semialgebraic set of positive codimension. A Nash (i.e., C^∞ and semialgebraic) version of this result was given by Coste and Shiota who proved that any Nash map $f: X \to \mathbb{R}^k$ on a closed Nash manifold $X \subset \mathbb{R}^n$ is locally Nash trivial outside some Nash subset.

It is an important open problem to have a precise description of the bifurcation set, that is, the smallest set B such that the restriction $f: X \setminus f^{-1}(B) \to$

$\mathbb{R}^k \setminus B$ is locally trivial. For a C^r ($r \geq 2$) semialgebraic mapping $f: \mathbb{R}^n \to \mathbb{R}^k$, Kurdyka, Orro and Simon showed that f admits local C^r (not necessarily semialgebraic) trivializations over the complement of the set $K_0(f) \cup K_\infty(f)$. (In particular, the bifurcation set is contained in $K_0(f) \cup K_\infty(f)$.) The main result of [A4] (Theorem 1.2) asserts that a Nash mapping $f: X \to \mathbb{R}^k$ on a closed Nash manifold $X \subset \mathbb{R}^n$ is locally Nash trivial outside the set $K(f)$ of generalized critical values.

· In [A5], A. Valette proved a generalized version of the Łojasiewicz inequality about the gradient of a C^1 globally subanalytic function. The famous Łojasiewicz inequality says that, given a C^1 globally subanalytic function-germ $f: (X, a) \to \mathbb{R}$, with X a globally subanalytic C^1 submanifold of $\mathbb{R}^n$ and $a \in X$, there exists a neighbourhood U of a in X, a constant C and a rational number $\theta \in [0, 1)$ such that for all $x \in U$,

$$|f(x) - f(a)|^\theta \leq C|\nabla_x f|,$$

where $\nabla_x f$ is the gradient of f at x.

In [A5], A. Valette explored the case where the point a does not belong to X but lies in its closure. (In particular, a may be a singular point of the closure of X.) More precisely, if $f: X \to \mathbb{R}$ is a C^1 globally subanalytic function on a bounded C^1 submanifold $X \subset \mathbb{R}^n$, then, by the Sard theorem for the asymptotic critical values which we have discussed above (Theorem 4.3 of [A3]), the set of asymptotic critical values of f is finite. In Theorem 1.3 of [A5], A. Valette then proved that if $K(f) \neq \emptyset$, then there exists a rational number $\theta \in (0, 1)$ and a constant $C > 0$ such that for all $x \in X$,

$$dist(f(x), K(f))^\theta \leq C|\nabla_x f|,$$

where *dist* is the Euclidean distance.

As a corollary, A. Valette also obtained a Łojasiewicz type inequality for nonsmooth functions which improves a previous result of Bolte, Daniilidis and Lewis. She showed that if $f: \mathbb{R}^n \to [-\infty, +\infty]$ is a globally subanalytic function, then the set L_f of so-called "asymptotic subcritical values" of f (Definition 2.5) is finite (Proposition 2.7), and if furthermore this set is not empty, then there exists a rational number $\theta \in (0, 1)$ such that the ratio

$$\frac{dist(f(x), L_f)^\theta}{m_f(x)}$$

is locally bounded near every point of $\mathbb{R}^n$ (Theorem 2.9), where $m_f(x)$ denotes so-called "nonsmooth slope" of f at $x \in \mathbb{R}^n$ (Definition 2.1).

· In the paper [A6], Kocel-Cynk, Pawłucki and A. Valette proposed a new proof of the definability of Hausdorff limits (first proved by Bröcker, Marker and Steinhorn, Pillay, van den Dries, and Lion and Speissegger). To give a precise statement of this result, let us consider the set $\mathcal{K}_n$ of all nonempty compact subsets of $\mathbb{R}^n$, equipped with the Hausdorff metric, and let us fix an o-minimal structure expanding the field of real numbers. Hereafter, "definable" means definable in this o-minimal structure. By definition, a subset $\mathcal{A} \subset \mathcal{K}_n$ is called definable if there exists a definable subset $T \subset \mathbb{R}^k$ (for some k) and a definable subset $A \subset \mathbb{R}^n \times T$ such that $\mathcal{A} = \{A_t \mid t \in T\}$, where $A_t := \{x \in \mathbb{R}^n \mid (x,t) \in A\}$. The result says that the Hausdorff closure of any definable subset $\mathcal{A}$ of $\mathcal{K}_n$ is definable (Corollary 1). In particular, the Hausdorff limit of a convergent sequence of subsets of a definable family $\mathcal{A} \subset \mathcal{K}_n$ is definable (Corollary 2). Actually, these are corollaries of the main theorem of [A6] which says that if T is a definable bounded subset of $\mathbb{R}^k$ and A is a definable bounded subset of $\mathbb{R}^n \times T$ such that all the fibers A_t ($t \in T$) are nonempty compact subsets of $\mathbb{R}^n$, then there exists a definable bounded subset $S \subset \mathbb{R}^k$ and a definable Lipschitz bijection $\varphi\colon S \to T$ such that the mapping $s \in S \mapsto A_{\varphi(s)} \in \mathcal{K}_n$ is Lipschitz, and hence, extends in a unique way by continuity to $\overline{S}$.

The techniques developed in [A6] turned out to be useful not only to prove the theorem above but also for other purposes. Especially, Proposition 1 (about Lipschitz parametrizations of definable families of Lipschitz functions) was used by G. Valette and N. Nguyen to improve the o-minimal preparation theorem of van den Dries and Speissegger, and as a consequence, to establish the existence of Lipschitz stratifications in the sense of Mostowski for sets which are definable in a polynomially bounded o-minimal structure expending the real field.

In the paper [A7], still developing the techniques of [A6], Kocel-Cynk, Pawłucki and A. Valette gave, in the case of an arbitrary o-minimal structure expanding a real closed field R, a new proof of so-called "uniform C^p-parametrization theorem" due to Yomdin and Gromov in the semialgebraic case. Another proof, by different methods, in an arbitrary o-minimal structure, was given by Pila and Wilkie.

*

Apart from the series of articles [A1]-[A7] which serves as a basis for the Habilitation procedure, A. Valette is also the author of another collection of works (presented as "Other Scientific Achievements" in the "Summary of Professional Accomplishments") consisting of 5 papers [B1]-[B5], and 3 preprints [C1]-[C3] (not available in the application file).

· In the series of papers [B1, B3, B4], A. Valette investigated the set J_f of points at which a given polynomial mapping f is not proper (so-called "asymptotic set" or "Jelonek set" of f). The study of this set is important and directly related to the Jacobian conjecture, which reduces to show that the Jelonek set of a polynomial mapping $f: \mathbb{C}^n \to \mathbb{C}^n$ with nonzero constant Jacobian is empty.

The starting point is a theorem of Jelonek which says that if $f: \mathbb{C}^n \to \mathbb{C}^n$ is a dominant polynomial mapping, then the set J_f is either empty or a $\mathbb{C}$-uniruled hypersurface (i.e., it is dominated by a cylinder). Later, Jelonek gave versions of this result for polynomial mappings between complex affine varieties, and also investigated the case of real polynomial mappings.

In the case of an arbitrary algebraically closed field k (of any characteristic), Jelonek showed that the set J'_f of points at which a generically finite polynomial mapping $f: k^n \to k^m$ is not finite is either empty or a variety of pure dimension $n-1$, and in Theorem 4.1 of [B3], A. Valette proved that this set is also k-uniruled. The notion of finite map is the natural algebraic counterpart of the notion of proper map. In the complex case, the set J'_f of points at which a polynomial mapping f is not finite coincides with the Jelonek set J_f of points at which f is not proper, and hence, is also referred to as the Jelonek set.

In [B1] (Theorem 4), A. Valette also gave a method to compute effectively the ideal of the set J'_f of a given dominant polynomial mapping $f: X \to Y$ betwen affine irreducible varieties $X \subset k^n$ and $Y \subset k^m$.

In the real case, the situation is more complicated, and so far there is no satisfactory such a general method. In [B4], however, A. Valette presented a generically effective method to determine the Jelonek set J_f of a given polynomial mapping $f: \mathbb{R}^2 \to \mathbb{R}^2$ with finite fibers. She showed that in this case J_f is equal to the support of an algebraically constructible function which can be generically obtained effectively (Theorem 3.3). As an important corollary of Theorem 3.3, she obtained that if the number of points in each fiber of f over some open set $U \subset \mathbb{R}^2$ is constant, then f is proper at U (Corollary 3.4).

· The paper [B2] is on the Noether exponent. The Noether exponent of an ideal $I \subset k[X_1, \ldots, X_n]$ (k algebraically closed) is the smallest number μ such that $(\operatorname{rad} I)^\mu \subset I$. An estimate of this exponent was obtained by Kollár. Later, using simpler methods, several authors contributed to this problem, especially in the case where the ideal I has no embedded primary components. In Theorem 7 of [B2], A. Valette obtained in this special case a sharper estimate.

· Finally, in the paper [B5], by Bilski, Kucharz, G. & A. Valette, it is proved that any pre-algebraic $\mathbb{F}$-vector bundle ($\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) on an affine real algebraic variety X can be made algebraic after finitely many blowups. More precisely, if ξ is a pre-algebraic $\mathbb{F}$-vector bundle on X, then there exists a regular birational map $\pi: X' \to X$ which is the composition of a finite number of blowups with nonsingular centers (so-called "multiblowup") such that X' is nonsingular and the pullback $\mathbb{F}$-vector bundle $\pi^*\xi$ on X' is algebraic (Theorem 1.1). If furthermore X is nonsingular, then there exists a Zariski closed subset $Z \subset X$ of codimension ≥ 2 such that the restriction $\xi|_{X \setminus Z}$ is an algebraic $\mathbb{F}$-vector bundle on $X \setminus Z$ (Corollary 1.2). In particular, any pre-algebraic $\mathbb{F}$-vector bundle on an 1-dimensional such a nonsingular variety X is algebraic (Corollary 1.3). The "regulous" maps introduced by Fichou, Huisman, Mangolte and Monnier play an essential role in the proof.

As an application, Bilski, Kucharz, G. & A. Valette also proved that if X is a compact nonsingular affine real algebraic variety, then the Stiefel-Whitney classes of any pre-algebraic $\mathbb{R}$-vector bundle on X are algebraic (Theorem 3.2). This extends to pre-algebraic $\mathbb{R}$-vector bundles a well known property of algebraic $\mathbb{R}$-vector bundles. Another well known fact about algebraic vector bundles is that the Chern classes of any algebraic $\mathbb{C}$-vector bundle on X and the Pontryagin classes of any algebraic $\mathbb{R}$-vector bundle on X are $\mathbb{C}$-algebraic. Examples showing that this property cannot be extended to pre-algebraic vector bundles are given in Example 5.2 of [B5]. However, it is proved, in Theorem 5.7, that the Chern classes of any pre-algebraic $\mathbb{C}$-vector bundle on X and the Pontryagin classes of any pre-algebraic $\mathbb{R}$-vector bundle on X are blow-$\mathbb{C}$-algebraic (i.e., they can be made $\mathbb{C}$-algebraic by pulling them back by a suitable multiblowup).

*

In conclusion, the articles presented by Dr. Anna Valette, both for the Habilitation procedure and as "Other Scientific Achievements," represent an important and original contribution to various and difficult areas of complex algebraic geometry as well as real algebraic and analytic geometry. The techniques developed in these articles clearly show a thorough knowledge of A. Valette in these domains of mathematics.

In my opinion, the material presented satisfies all the conditions required for Dr. Anna Valette to receive the Habilitation of the Jagiellonian University.

C. Eyral

Dr. hab. Christophe Eyral
Institute of Mathematics
Polish Academy of Sciences
Warsaw, 15 March 2019