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Report on Dr. Michal Kapustka's scientific achievements on "Calabi-Yau threefolds and Mukai varieties"

Dr. Michal Kapustka's scientific achievements are classified into the following three major subjects: 1) Calabi-Yau threefolds and mirror symmetry, 2) Calabi-Yau threefolds in $\mathbb{P}^6$, and 3) Mukai varieties. I see in his published papers that Dr. Kapustka has obtained a lot of interesting mathematical results on each of these subjects. They have independent mathematical importance from others, however, in my report I would like to review them from recent developments in mirror symmetry of Calabi-Yau threefolds. From this viewpoint, I recognize many important contributions by Dr. Kapustka, e.g., his general and explicit constructions of the so-called Pfaffian Calabi-Yau threefolds and also other interesting non-complete intersection Calabi-Yau threefolds. I evaluate highly his achievements, and also expect future developments related to them. Based on them, there is no doubt for me that his achievements deserve a habilitation title.

Mirror symmetry of Calabi-Yau manifolds was discovered in early 90's in theoretical physics. Soon after its discovery, it was recognized that the symmetry can be described systematically for Calabi-Yau complete intersections in toric varieties, and in these cases, we can go further into more concrete predictions about the invariants of Calabi-Yau manifolds such as Gromov-Witten invariants. Although an ultimate mathematical definition of mirror symmetry is still missing, we have now large accumulations of techniques for describing the symmetry and also applications of it to Gromov-Witten theory, for example, for concrete examples of complete intersection Calabi-Yau manifolds in toric varieties. Along with these surprising applications, up to now, several versions of proposals toward the mathematical definition of mirror symmetry have been appeared. Among them, homological mirror symmetry due to Kontsevich is one of the most promising and attractive one, and it has put a strong motivation to investigate Calabi-Yau threefolds which are not given as complete intersections in toric varieties. Such motivation mainly comes from our interests in describing the derived categories of coherent sheaves on Calabi-Yau manifolds, which in the case of two dimensional Calabi-Yau manifolds, i.e. K3 surfaces, has been studied extensively by Mukai in relation to the moduli spaces of stable vector bundles on K3 surfaces. In particular, we know that several homogeneous spaces appear in Mukaki's classification of the prime Fano varieties.

Having these historical backgrounds of mirror symmetry in mind, I would like to describe some details about Dr. Kapustka's contributions to the three major topics 1) to 3) above.

1) Calabi-Yau threefolds and mirror symmetry: Classification of Calabi-Yau threefolds up to deformations or birational transformations is one of the important problems in higher dimensional algebraic geometry (and even for string theory in theoretical physics). For such classification, similarly to the case of Fano threefolds, it is natural to focus on Calabi-Yau manifolds with Picard number $\rho(X) = 1$. These Calabi-Yau manifolds of $\rho(X) = 1$ also play special roles in verifying mathematical predictions from mirror symmetry. This is because the corresponding mirror families, if we have, will be defined over one-dimensional deformation spaces, and we can expect Picard-Fuchs differential equations defined over $\mathbb{P}^1$ for the period integrals of holomorphic three forms. This simplifies the verifications of many important predictions from mirror symmetry (such as Gromov-Witten invariants) which are expressed in terms of the solutions of Picard-Fuchs equations.

Moreover, as observed in several examples, we can read the so-called Fourier-Mukai partners of X from certain characteristic properties of the Picard-Fuchs differential equations near boundary points. A famous example is the case of Calabi-Yau threefold given by linear sections of $\mathrm{G}(2,7)$ and the Pfaffian Calabi-Yau threefold in $\mathbb{P}^6$ given by $\mathrm{Pf}(7)$ of 7×7 skew symmetric matrix. Recently it has been found that a certain determinantal variety also has a non-trivial Fourier-Mukai partner. It is now expected that these phenomena of (non-trivial) Fourier-Mukai partners may be observed more if we have classification of Calabi-Yau manifolds with Picard number $\rho(X) = 1$. It is also expected recently that these phenomena are closely related to the classical projective duality which naturally arises in the construction of Calabi-Yau threefolds of Picard number one.

- In the paper [H7][1], Dr. Kapustka has proposed to use Kustin-Muller unprojection to generate Calabi-Yau threefolds with Picard number one systematically starting with del Pezzo surfaces of various ways of their realizations with varying codimensions in (weighted) projective spaces. From his construction, he has not just reproduced systematically known examples of Calabi-Yau threefolds with $\rho(X) = 1$ but also found new Pfaffian Calabi-Yau threefolds in weighted projective spaces $\mathbb{P}(1^4, 2^3)$, $\mathbb{P}(1^5, 2^2)$ and $\mathbb{P}(1^6, 2)$. One of the advantages of his construction is that it gives us explicit defining equations of Calabi-Yau threefolds, which we need in order to find their mirror families.

Constructing a mirror family to a given Calabi-Yau threefold is still open problem unless Calabi-Yau threefold is given by a complete intersection in a toric variety. In the paper [HP1], it has been shown that, for the newly-found Pfaffian Calabi-Yau threefolds, we can obtain their mirror families by the method of toric degeneration by considering cones over the Pfaffian Calabi-Yau threefolds and their linear sections. It seems that the cone construction proposed there will be an efficient method to have mirror family

[1] Here and in what follows, reference numbers are those given in "Scientific Report".

for a wider class of Calabi-Yau manifolds. Dr. Kapustka's new examples are giving us intuitions for the mirror symmetry beyond the standard complete intersections in toric varieties.

• In the paper [H8], it has been shown that Kustin-Muller unprojection also gives us a systematic way to generate a sequence of determinantal Calabi-Yau threefolds if we start with equations of determinantal del Pezzo surfaces. Determinantal Calabi-Yau threefolds obtained from the del Pezzo surfaces of degree 5 and 8, respectively, are Pfaffian Calabi-Yau threefold Pf(7) (whose mirror symmetry was studied by Rødland [138] in 1998) and its symmetric analogue with 5×5 symmetric matrices (whose mirror symmetry has also been studied [68] in 2011). It has been recognized recently that the classical projective geometry behind these Calabi-Yau threefolds provides nice examples for a modern proposal called Homological Projective Duality (HPD) due to Kuznetsov. As a consequence of HPD, it is now known that both the determinantal Calabi-Yau threefolds for the del Pezzo surfaces of degree 5 and 8 have their (non-trivial) Fourier-Mukai partners. Also, the determinantal Calabi-Yau threefold corresponding to the del Pezzo surface degree 6 is attracting some attention in theoretical physics since this provides a non-trivial example for which a recent technique of Gauged Linear Sigma Model can be applied to calculate, for example, Gromov-Witten invariants without its mirror family. What is puzzling for this Calabi-Yau threefold, however, is that naive applications of known methods for the construction of the mirror Calabi-Yau manifold do not work, and a new idea seems to be required. Looking back the history of mirror symmetry, there is no doubt in saying that these examples are driving forces for new developments.

2) Calabi-Yau threefolds in $\mathbb{P}^6$: Dr. Kapustka's contributions to this topic are motivated by the classification problem introduced by Prof. Okonek. His results on the classification have their own mathematical importance, but I would like to note that they are given together with the method giving equations for Calabi-Yau manifolds. Having such equations are very important to find mirror Calabi-Yau manifolds, whose constructions are not clear beyond complete intersections in toric varieties.

3) Mukai varieties: Historically the study of Mukai varieties has started purely from mathematical motivation to classify Fano threefolds. However, interestingly, this is closely related to the topic 1) under the key words of Fourier-Mukai partners and HPD as mentioned above. Dr. Kapustka has an interesting and important contribution in this topics related to Mukai's Fano threefold M_{10} and its related geometry of Fourier-Mukai partners.

Mukai dual of a K3 surface is another K3 surface which appear as a moduli space of stable vector bundles on the K3 surface. The most famous example is a K3 surface S_{12} in M_7 for which the Mukai dual is given by the moduli space $M_{S_{12}}(2, H, 3)$. In the paper [H5], Dr. Kapustka and Ranestad have given an explicit description of the Mukai dual $M_{S_{18}}(3, H, 3)$ for a K3 surface in M_{10}. In the former case, since the moduli space is fine and there is a Fourier-Mukai functor, the two K3 surfaces are Fourier-Mukai partners to each other. For the latter case, on the other hand, we need to consider the "twisted" Fourier-Mukai partners, since the moduli space is not fine. Owing to the results by Kapustka and Ranestad, we now know about the detailed geometries for both K3 surfaces. I expect some more explicit geometries if we take mirror symmetry into account for both cases.

Also, when we turn our attentions to Calabi-Yau threefolds, the above examples of K3 surfaces naturally motivate us to construct Calabi-Yau threefolds with Picard number one and study their mirror symmetry, for example, using the Picard-Fuchs differential equations.

As above, Dr. Kapustka's mathematical contributions to the three major topics 1) to 3) are closely related to each other from the viewpoint of Calabi-Yau threefolds. In particular, I recognize in his works many "interesting" Calabi-Yau threefolds of non-complete intersections whose mirror families have not been studied well.

To fully understand or extract all mathematical implications contained in the conjecture of mirror symmetry, probably we need to be involved in more transcendental aspects of the symmetry, i.e., deformation theory, Hodge theory, and so on. In such cases, we will need to consider Calabi-Yau threefolds with higher Picard numbers than $\rho(X) = 1$. Even in that case, I do expect that his accumulations of interesting examples and systematic study of interesting classes of Calabi-Yau threefolds of $\rho(X) = 1$ will play important roles when we extend our scopes toward that.

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