REPORT ON M. SLAWOMIR DINEW'S DISSERTATION SUBMITTED AS A HABILITATION THESIS AT JAGIELLONIAN UNIVERSITY

PHILIPPE EYSSIDIEUX

I have known S. Dinew as a specialist of Complex Monge-Ampère equations from his PHD years around 2007. The way I got acquainted with his work was that in joint works with my coauthors Berman, Boucksom, Guedj and Zeriahi, we needed to use one of his results crucially.

He has authored 19 papers mainly on complex Monge-Ampère and Hessian equations with various coauthors, making each time a notable contribution. The journals are well-known and well-regarded international journals either specialized like Journal of Functional Analysis or aiming at a more general mathematical readership like Mathematische Annalen, Mathematische Zeitschrift, Advances in Mathematics and American Journal of Mathematics.

My report will survey what seem to me the most important contributions from the research articles presented in the dissertation.

In what follows, M will be a complex manifold of dimension $n \geq 2$, in fact one is mostly interested in cases $M = \Omega$ a stricly pseudoconvex domain (local setting) or $M = X$ a compact Kähler manifold (global setting), ω will be a Kähler form on X, f a real valued function on M and m will be an integer such that $2 \leq m < n$.

1. Complex Monge-Ampère Equations

This topic is extremely important in modern Complex Analysis, with applications in the theory of holomorphy enveloppes and to Kähler Geometry since Kähler-Einstein metrics and geodesics of the space of Kähler metrics are governed by such equations and have been studied by crowds of mathematicians including people of the highest calibre such as Calabi, Yau, Bedford, Taylor, Donaldson, Semmes, Mabuchi, Tian, Chen ... The real theory has been developped by Pogorelov for its applications to convex surfaces and serves as a model case of Fully nonlinear elliptic equations. So, it is important to have as sharp as possible regularity results.

1.1. **Regularity of Sobolev solutions.** The first paper surveyed in the dissertation is a collaboration with Z. Blocki proving the interior $C^{1,\bar{1}}$-estimate for the complex Monge-Ampère equation:

$$\det\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right) = f$$

with $0 < f \in C^{1,\bar{1}}$ under the hypothesis $u \in W^{2,p}$ for $p > \frac{n}{n-1}$. There are counterexamples for smaller p. This optimal result has a rather subtle (and short) proof which seems to be essentially the best possible.

Date: November 16, 2017.

1.2. **Regularity of Hölder solutions.** The second paper surveyed in the dissertation is a collaboration with Xi Zhang and Xiangwen Zhang proving the interior $C^{2,\alpha}$-estimate for the complex Monge-Ampère equation:

$$\det(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}) = f$$

with $0 < f \in C^\alpha$ under the hypothesis $u \in C^{1,1}$. This has been relaxed to $u \in C^{1,\bar{1}}$ later by Y. Wang.

1.3. **Hölder continuity of solutions to global Monge-Ampère equations.** A famous result of Kolodziej states that the solutions to on X

$$(\omega + dd^c\phi)^n = f\omega^n,$$

with $0 \leq f \in L^p$ are Hölder continuous with an exponent α that depends on p, n but may depend on the geometry of (X, ω). This is a very unpleasant circumstance since in the case of $\mathbb{P}^n$ the proof is much simpler (it uses the fact that the the Fubini Study metric is homogenous) and gives an exponent depending only on p, n.

The dissertation presents a very interesting article of the author where he insightfully uses a regularisation technique of Demailly to achieves independance of the geometry, more precisely:

$$\alpha = \frac{1}{np/(p-1) + 1 + \epsilon},$$

under a positivity assumption on the bissectional curvature that has not been previously considered as far as I know. This would work for $X = (\mathbb{P}^{n-1} \times C, \omega_{FS} + \omega_C)$ for instance. Compared to the homogenous case, one gets an extra non-homogenous dimension. This kind of result is extremely intricate and I am impressed by the achievement. The proof elaborates in a very non trivial way a rather important paper of Guedj, Zeriahi with Kolodziej.

This was an important step to the independance of the geometry of the Hölder exponent whixh was finally established in a many authors' paper that S. Dinew coauthored with the aforementioned people, Demailly and Pham. This very satisfying article was published in the Journal of the European Mathematical Society. It is not surveyed here for bibliographical reasons but this is certainly the final word on this important problem.

2. Complex Hessian Equations

This topic has no known applications and has consequently been less studied but I believe that (a variant yet to be formulated of) these equations could be immensely helpful in applications to Algebraic Geometry. Nevertheless it is very interesting per se and gives rise to many challenging questions.

2.1. **Local case.** The subsolutions of the complex Hessian equation

$$(dd^c\phi)^m \wedge \omega^{n-m} = 0$$

on $\mathbb{C}^n$ are called m-subharmonic functions. They share sufficiently many formal features of plurisubharmonic functions so that one can develop a nonlinear potential theory analogous to the Bedford-Taylor potential theory. This was done by S.Y. Li and Z. Blocki.

A priori estimates for the complex Hessian equation are the topic of one of the articles surveyed in the dissertation. This collaboration with S. Kolodziej develops methods to estimate solutions using more advanced potential theoretic tools in the flavor of S. Kolodziej famous L^∞-estimate for the complex Monge-Ampère equations. They obtain L^p_{loc} integrability for m-sh functions with relatively compact level sets for $p < \frac{nm}{n-m}$ and L^∞ estimates for the Dirichlet problem on Ω (or its global counterpart) for the inhomogenous equation if the right hand side is in L^q for $q < \frac{n}{n-m}$. The method also yields stability results in a standard way.

The techniques employed here are not completely surprising, Kolodziej-type Volume/Capacity inequalities have been recognized and used over and over as a most important tool for non linear fully elliptic equations, even in degenerate cases. However, in each specific case, there is something subtle to be done.

The main open question that remains unsettled in this field is Blocki's conjecture claiming L^p_{loc} integrability of m-sh functions for $p < \frac{nm}{n-m}$, without any assumption on the compacity of the level sets.

I guess the second main motivation for this study was the global case.

2.2. **Global case.** The most impressive result in this survey is a collaboration with S. Kolodziej proving that the global Hessian equation on X

$$(\omega + dd^c\phi)^m \wedge \omega^{n-m} = f\omega^n,$$

with $f > 0$ and smooth, has a unique (up to a constant) smooth solution. This question attracted the attention of several groups of mathematicians and was quite a competitive research problem. It has been accepted in American Journal of Mathematics, a prestigious journal of the field. A well known work by Hou-Ma-Wu reduced the problem to a gradient estimate, which turned out to be very elusive. The authors use a renormalisation technique to reduce to a Liouville type theorem for Lipschitz m-subharmonic functions on $\mathbb{C}^n$ solving the homogenous complex Hessian equation (uniformly bounded such functions ψ are constant). It should be noted that this statement fails to hold for general Lipschitz continuous m-subharmonic functions. In turn the Liouville type theorem is established by an extremely ingenious method that has actually no counterpart I know of using an analysis of the quadratic means on balls of ψ. To a geometrically inclined referee, there is some analytical black magic taking place here! Unsurprisingly, it also relies on the local theory surveyed above.

3. Conclusion

The submitted dissertation is a nice piece of scolarship and gives an interesting survey of the fundamental problems S. Dinew has been working on. The main ideas are explained in a pedagogical way so that it's easy to see what were the technical hurdles to be overcome. The new proof techniques invented by S. Dinew and his coauthors are highly original and inventive, mixing in a skillful way techniques from the analysis of the real Monge-Ampère and Hessian equations from Caffarelli's school and specific techniques from complex analysis.

I am therefore convinced that S. Dinew's work is of the highest quality and that he is one of the leading researchers of his generation worldwide in the field of Partial Differential Equations in Complex Analysis and their applications.

It qualifies him for supervising graduate students and holding a Professor position in a good university.

I support most strongly and without any reservation his application for being granted the habilitation title from Jagiellonian University.

Philippe Eyssidieux.
Professeur.
Université Grenoble-Alpes.