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**Report on the PhD dissertation by Shengda Zeng entitled**

**"Constrained variational and hemivariational inequalities with applications"**

This dissertation consists of 5 papers of Shengda Zeng published jointly with the PhD supervisor as well as with other authors. Below, I shall briefly describe and discuss main results from those papers emphasizing mathematical methods used in their proofs.

- (I) S. Migórski, S.D. Zeng, *Hyperbolic hemivariational inequalities controlled by evolution equations with application to adhesive contact model.* Nonlinear Anal. Real World Appl. 43 (2018), 121–143.

Here, the authors consider the second order hyperbolic equation

(1) $$u'' + Au' + bu + M^*\partial J(\beta, Mu) = f$$

coupled with the ordinary differential equation

(2) $$\beta' = F(t, Mu, \beta)$$

supplemented with usual initial conditions. The functional $J$ is locally Lipschitz, thus $\partial J$ denotes the generalized Clarke subgradient operator and the hyperbolic equation (1) has to be understood as the differential inclusion, or equivalently, as a hyperbolic hemivariational inequality. In the paper, the authors show the existence of solutions to system (1)-(2) supplemented with initial conditions by discretizing it in time variable (so-called Rothe method) and proving the existence of solutions to an approximating problem by using a well-known result for pseudo-monotone operators. Section 2 of this paper contains a long list assumptions which permit to apply this procedure in rather usual way: the operator $A$ is pseudomonotone, $B$ is linear, bounded, symmetric, and coercive, the nonlinearity in equation (2) globally Lipschitz continuous. Section 3 contains an application of obtained result to a frictional contact problem.

This is a perfectly well-written paper: its notation is explained in detail, results used in this work are recalled with precise references, the proofs are as detailed as possible. Although this is a long paper and obtained results look very complicated, an idea behind this work is stand and expected due to very strong assumptions. The discretization stated in Problem 10 of this work is standard and always appear in studies of second order hyperbolic equations. The existence of the numerical scheme is shown by a direct application of the classical theory for pseudomonotone operators and the convergence is obtained by usual compactness method. It seems that system (1)-(2) is a direct generalization of the frictional contact problem (Problem 16 in this work), so the existence of its solutions results immediately.

Some parts of this work, according to the reviewer, are too detailed. For example, Theorem 4 stated in this work, on the existence of solution to the Cauchy problem for an ordinary differential equation with the Lipschitz nonlinearity is known to every student of Mathematics. A similar comments concerns Lemma 8 – this is an immediate consequence of the Gronwall lemma and there is no need to recall the proof. The imbedding in Theorem 15 are completely standard and could be omitted in this paper. Proposition 3 is called the Aubin-Lions lemma and it is a main compactness tool in studies of partial differential equations. This is well-known and old result and I am surprised that the paper refers to recent work [24] by P. Kalita for its proofs.

- (II) S. Migórski, S.D. Zeng, *Penalty and regularization method for variational-hemivariational inequalities with application to frictional contact*. ZAMM Z. Angew. Math. Mech. 98 (2018), no. 8, 1503–1512.

In this work, a certain class of variational-hemivariational inequalities of elliptic type is considered, where the existence of solutions have been shown in other papers. The main goal of this work is show that a certain regularization of that inequality is well-posed and solutions of those regularizations converge towards an original problem. The proof involves usual compactness methods and typical estimates which appear in the theory for pseudo-monotone operators. This paper consists of 18 pages, but in fact, the main results are proved on 4 pages (Lemma2 11-14). The reminder of this work contains long explanations of notation used in this work. Applications to stability of a contact problem is direct and a rather simple idea is hidden behind a complicated notation.

- (III) S. Migórski, S.D. Zeng, *A class of generalized evolutionary problems driven by variational inequalities and fractional operators.* Set-Valued and Variational Analysis (2018), 22 pp.

In this work, a system similar to the one in (1)-(2) is considered, now, with $A = 0$ and there the second derivative $u''$ is replaced by a Riemann-Liouville fractional derivative of $u$. The existence of solutions is shown by using the Bohnenblust-Karlin fixed point theorem which is a generalization of the Schauder fixed point theorem to multivalued mappings.

- (IV) S. Migórski, A.A. Khan, S.D. Zeng, *Inverse problems for nonlinear quasi-variational inequalities with an application to implicit obstacle problems of p-Laplacian type.* Inverse Problems 35 (2019), no. 3, 035004, 14 pp.

Here, a theory of pseudomonotone mappings is used to study properties of solutions to certain inequalities related to an obstacle problem of *p*-Laplacian type. As in other papers from this dissertation, several well-known theorems used in the proof have been recalled. For example, Appendix contains a well-known definition of hemicontinuous monotone operators and a definition of a convex lower semicontinuous function. Such a detailed presentation is appreciated in a PhD dissertation but it could be skipped in a research paper addressed to specialists.

- (V) S. Migórski, M. Sofonea, S.D. Zeng, *Well-posedness of history-dependent sweeping processes*. SIAM J. Math. Anal. 51 (2019), no. 2, 1082–1107.

Here, an existence of solutions to a certain first order differential inclusion has been shown by using a time discretization method, the theory of monotone operators, and usual compactness methods. The obtained result is applied to a history-dependent viscoelastic contact problem. I find this paper the most interesting among all papers from this dissertation because the authors have to deal with several problems caused by a new setting of hemi-variational inequalities.

Let me summarize my opinion on results contained in the papers by Shengda Zend from his PhD dissertation.

- All papers a very well and detailed written. Notation – even if it is completely standard – is explained in great detail. Definitions of well-known mathematical objects and classical results used in a paper are always recalled, precisely. According to the referee, this is a too detailed presentation; for example, there is no need to recall a definition of a monotone or pseudomonotone operator in each paper.
- Ideas used in proofs are classical and well-known – they are not surprising, at all. Those classical arguments are hidden behind an involve formulation of considered problems.
- Almost always, the main difficulty of obtained results consist in extending classical variational methods to the case of variational inclusions. Often, such an extension is direct and completely natural.
- My comments above can be also applied to all 25 already-published papers of Shengda Zeng which is an enormous amount of papers for a PhD student. A “production” of so many papers in so-short time was possible because results from most of them are direct extension of previous work. Here, I would like to recommend the Author of the PhD dissertation to concentrate more on a quality of his papers and consider more ambitious problem rater them publishing a large amount of papers with simple extensions of known results.

*Beside my critical comments above, I am completely convinced that this is a good PhD dissertation and I strongly recommend to award the PhD title to Shengda Zend without any hesitation.*

G. Karch