Report on Ph.D. thesis

Algorytmy ścisłego całkowania równań wariacyjnych i ich zastosowania do badania bifurkacji rozwiązań okresowych w Kołowym Ograniczonym Problemie Trzech Ciał

by

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The dissertation consists of two main parts. In the first part the author proposed an implicit algorithm for validated enclosures of the solutions to the set of initial-value problems

$$\begin{cases} \dot{x}(t) = f(x(t)), & x(0) \in [x_0], \\ \dot{V}(t) = Df(x(t)) \cdot V(t), & V(0) \in [V_0], \end{cases}$$

where $[x_0] \subset R^n$ and $[V_0] \subset R^{n^2}$ are sets of initial conditions. All computations related to this algorithm are performed in interval arithmetic and at one step generate the sets $[x_1] \subset R^n$ and $[V_1] \subset R^{n^2}$ such that $x(h) \in [x_1]$ and $V(h) \in [V_1]$.

This algorithm consists of the prediction step based on the C^1-Lohner algorithm due to Zgliczyński [3] and the new C^1-HO implicit corrector step based on Hermite-Obreschkov interpolation [1] of the high order. It is constructed in such a way that the sets $([x_{k+1}], [V])$ computed by the algorithm satisfy

$$\begin{cases} \varphi(h_k, [x_k]) \subset [x_{k+1}], \\ \psi(h_k, [x_k], I) \subset [V], \end{cases}$$

as demonstrated in Lemma 6 on page 36. Moreover, this algorithm returns error bounds that are no worse, and usually better, than the error bounds produced by the C^1-Lohner algorithm. This is accomplished by intersecting the results obtained by the corrector step (C^1-HO) with that obtained by the predictor step (C^1-Lohner).

The author compares also the costs of one step of C^1-Lohner and C^1-HO algorithms assuming that they have the same order m. The asymptotic formulas for these costs, denoted by C_{LO} and C_{HO}, are derived, and it was demonstrated that they satisfy the relation

$$\frac{C_{HO}}{C_{LO}} \approx \frac{9m^2 + 38m + 132}{8m^2 + 3m + 100},$$

which approaches 9/8 for high orders. The author then argues that this somewhat higher extra cost of C^1-HO algorithm is compensated by the larger time steps the new algorithm can use without losing accuracy, because of tighter error bounds.

The author compares also the performance of C^1-Lohner and C^1-HO algorithms on four test problems: the Lorentz system (4.8), the Hénon-Heiles system (4.9), the Planar Circular Restricted Three Body Problem (PCR3BP) (4.10), and the Galerkin projection of infinite dimensional system of ODEs (4.11) onto $(a_1, a_2, \ldots, a_n)$ variables appearing in (4.11). The graphs of $S(t)$ defined by

$$S(t) = \log_{10} \mathrm{diam}([V(t)])$$

versus time t are presented and they illustrate that $S(t)$ for C^1-HO algorithm are no greater that $S(t)$ for C^1-Lohner algorithm. They are usually smaller, in many cases, by two orders of magnitude.

As an application of the C^1-HO algorithm the author provided a new computer assisted proof for the existence of a compact, connected, invariant attractor $\mathcal{A}$ in the Rössler system of ODEs

$$\dot{x} = -y - z, \quad \dot{y} = x + by, \quad \dot{z} = b + z(x - a),$$

for some specific values of the parameters a and b ($a = 5.7$, $b = 0.2$).

The results of the first part of this Ph.D. thesis has been published in a recent paper [2] in a high quality journal in applied mathematics.

In the second part of the thesis (Chapter 5 and 6) the author investigates the existence of period-tupling and touch-and-go bifurcations for differential systems with time reversal symmetry. These bifurcations correspond to the normal form $f_\nu^k(x)$ defined by the formulas (5.1) and (5.2) in the thesis. It was demonstrated in Theorem 12 that under some conditions (assumption **C1** on page 61, assumptions **C2-C4** on page 67, and the relation (5.12) on page 69) the form $f_\nu^2(x)$ has touch-and-go bifurcation of order k at some point, and in Theorem 13 that if $k \geq 2$ is even and condition (5.14) is satisfied the form $f_\nu^2(x)$ has period-tupling bifurcation of order k at some point. The symmetry breaking bifurcations for differential systems with symmetries S (Definition 13) and R (Definition 14) are investigated in Theorem 14. In Theorems 15 and 16 the author proves that under assumptions **C1-C4** the bifurcation functions can be transformed to normal forms (formulas (5.20) and (5.24) int he thesis), which are easier to investigate.

The theory presented in Section 5.1 is then adapted in Section 5.2 to Hamiltonian systems (5.25). This leads to new and interesting results about touch-and-go bifurcations in Theorem 18, period-tupling bifurcations in Theorem 19, and symmetry breaking bifurcation in Theorem 20. The author presents also an algorithm for computation of approximations to bifurcation points in Section 5.3.

The applications of the theory developed in Section 5 are given in Section 6. In particular, the author proves that the Michelson system of ODEs

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = c^2 - y - \frac{1}{2}x^2,$$

which has time reversal symmetry, has period-tupling bifurcation (Theorem 21). Bifurcations of halo orbits (which are important in planning cosmic missions) in Circular Restricted Three Body Problem (CR3BP) are investigated in Theorem 22, 23, and 24.

Sections 5 and 6 contain a lot of new results which provide significant new insight into the properties of some systems of ODEs. In my opinion these results are clearly publishable, and this reviewer is puzzled that no publication plans for these results were announced in this thesis.

Some remarks

- The Section 4.1 is confusing. It is not clear why the polynomials $P_{p,q}(s)$ are introduced and the derivation of the formula (4.1) is not explained in a satisfactory way. Also no reference is made to the original work of Obreschkov [1]. These topics are explained somewhat better in [2], where also the reference [1] is cited.

- On page 78 the assumptions **HC2-HC4** are formulated for the Hamiltonian systems investigated in Section 5.2. There is no assumption **HC1**.

- Some of the results presented in Table 4.1 on page 59 seem to be incorrect. For example for $m = 22$ the reported time of computations corresponding to $tol = 10^{-16}$ is smaller than that reported for $tol = 10^{-10}$. Similar comments apply to some other entries in this table.

- On page 94 it is written: "Obliczenia, które wskazują na występowanie lokalnego minimum są przeprowadzone w sposób nieścisły, ale z bardzo wysoką precyzją (400 bitów mantysy) arytmetyki zmiennoprzecinkowej [44], metodą całkowania rzędu 80-ego oraz tolerancją 10^{-60}." What is the point of using such high precision in non-precise (nieścisłych) computations?

- At the end of the proof of Theorem 22 it is written: "Na koniec, korzystając z metod opisanych w [53, 108] sprawdzamy, że wszystkie kawałki wykresów h_i sklejają się w sposób gładki." This is not satisfactory. The proof should be self-contained and the details using techniques of [53, 108] should be provided. Similar comment applies to the proof of Theorem 23.

Summary statement

The results of the first part of the thesis has been already published in the journal Applied Mathematics and Computation, and the results obtained in the second part of the thesis provide a significant new insight into the properties of some differential systems, and should be submitted for publication. I believe this Ph.D. dissertation exceeds the criteria for granting Ph.D. degree in mathematics. Hence, I support granting Ph.D. degree with distinction (z wyróżnieniem) to Irmina Walawska.

References

[1] N. Obreschkoff, Neue Quadraturformeln, Abh. Preuss. Akad. Wiss. Math-Nat. Kl. (4)1940, 20.

[2] I. Walawska and D. Wilczak, An implicit algorithm for validated enclosures of the solutions to variational equations for ODEs, Applied Mathematics and Computation, 291(2016), 303–322.

[3] P. Zgliczyński, C^1-Lohner algorithm, Foundations of Computational Mathematics, 2(2002), 429–465.

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