To the Scientific Council of the Faculty of Mathematics and Computer Sciences, Uniwersytet Jagielloński w Krakowie

Review of the the PhD thesis of Mr. Pawel Bogdan entitled: "Computational study of polynomial automorphisms".

SCOPE OF THE DISSERTATION AND SIGNIFICANCE IN THE RESEARCH FIELD

The aim of the thesis is to study several computational aspects of polynomial automorphisms related with the longstanding and yet unsolved Jacobian conjecture. The current formulation of the Jacobian conjecture (originally stated by O.-H. Keller for $K = \mathbb{Z}$ and $n = 2$ in 1939 and often just called Keller's question) may be formulated in the following way: let K be a field of characteristic zero and let n be a positive integer. Consider a polynomial mapping $F = (F_1, \ldots, F_n) : K^n \to K^n$ such that the Jacobian determinant $J(F)$ is a non-zero constant. Then, F is a polynomial automorphism. That is, there exists another polynomial mapping $G = (G_1, \ldots, G_n) : K^n \to K^n$ such that $F \circ G = Id_{k^n}$ and $G \circ F = Id_{k^n}$. It is clear that the condition $J(F)$ to be a non-zero constant is a necessary condition for F to be a polynomial automorphism, so the conjecture states that the condition is also sufficient.

The simple formulation of the conjecture hides a very difficult question which appears often as one of the most important mathematical problems to be solved. It is, for instance, problem number 16 in Stephen Smale's 1998 list of Mathematical Problems for the Next Century. Certainly, there have been numerous attempts to solve the conjecture, but with very few general results and many false proofs or "almost" proofs (even recent attempts by well recognized specialists). Noteworthy it is yet open even for the original case by Keller. Besides, the study of the Jacobian conjecture is related with several other difficult problems. Hence it has been looked at from many different points of view and there exists a really vast literature on the subject. In relation to this doctoral thesis, one must point out that the conjecture has been proved affirmatively by S. S-S. Wang in 1980 for quadratic maps. In addition, A. V. Yagzhev (1980) and H. Bass, E. H. Connell and D. Wright (1982) have shown that if the conjecture holds for all $n \geq 2$ and all polynomial maps of degree at most three then it holds in general. Next, L. M. Drużkowski (1983) proved that it suffices to prove the conjecture for polynomial mappings where $F_i = X_i + H_i^3$ where H_i is a linear form for any $i = 1, \ldots, n$. The difficulty of these works is considerable.

A different approach to the Jacobian conjecture was done in 1973 in a very interesting paper by L. A. Campbell, relating the conjecture to an equivalent statement in tems of algebraic Galois theory. This equivalence has been recently extended to the setup of differential Galois theory by T. Crespo and Z. Hajto in 2011, as a consequence of their study on the equivalence of the two definitions of Picard-Vesiot extensions, by E. Kolchin in 1952 on one side and M. van der Put and M. Singer in 2003 on the other. In some sense, the outstanding part here is the fact that by using Kolchin's characterization of Picard-Vesiot extensions, one can check the validity of the Jacobian conjecture in terms of some generalized Wronskians. These are determinants in terms of the ordinary partial derivatives and the so called Nambu derivations, that one can construct by means of the Jacobian matrix of F. This leads to a computational way to study the Jacobian conjecture. Trying to understand more carefully the effective implementation of this approach seems to be the initial motivation of the present thesis.

The main result of the thesis is the description of an algorithm that may detect if a polynomial mapping F is an automorphism. In case that this algorithm finishes in a finite number of steps, one has that F is in fact an automorphism and as a byproduct of the algorithm itself one may also compute the inverse of F. This algorithm is rather easy to describe and do not involve complicated computations, even not any formulation in terms of algebraic Galois theory or differential Galois theory. This leads to the introduction of a new family of polynomial mappings, called Pascal finite, with some interesting properties. But it is also remarkable that if F is not Pascal finite one can still use the algorithm to formulate an equivalent condition for the invertibility of F in a finite number of terms of the polynomials constructed with the algorithm, as well as to compute the inverse of F in this case.

For such kind of problems as the Jacobian conjecture, any serious and rigorous approach to understand its difficulties can be considered meaningful. On the other side, given the high power of the current computers and the efficient design of the several available Computational Algebra Systems, to implement and test the same algorithm under different systems can provide useful information about the problem itself and the obstacles to be solved. So it is in this context that the significance of the contributions of this thesis to the Jacobian conjecture must be considered

Structure and contents

The thesis consists on a short Introduction, a first chapter with the preliminaries on differential Galois theory, a second one with the preliminaries on the Jacobian conjecture, a third chapter containing the inversion algorithm and its implementation, and a fourth chapter on the Pascal finite polynomial automorphisms. It also contains three appendices: Appendix A with the description of several common Computer Algebra Systems, B with several additional implementations of the algorithm, and C with some complicated formulas that appear in the thesis.

The Introduction is brief but enough to motivate the scope of the thesis. It also contains a concise description of the contents of the thesis as well as some notation. Chapter 1 consists on a short but complete introduction to the basics of differential Galois theory used in the thesis. In particular, Theorem 1.3.10 contains an improvement of the result by T. Crespo and H. Hajto cited above proving the equivalence of the different definitions of Picard-Vessiot extensions (Theorem 1 in [12], I use the same references as in the thesis). The improvement consists on bounding in a more efficient way the order of the differential operators appearing in the Wronskians to be checked if they belong to the base field. This improvement will be later used in other several parts of the thesis. Chapter 2 reviews the Jacobian Conjecture and its reductions relevant to the thesis. As said before, it suffices to prove the Jacobian Conjecture for cubics, for any n. More specifically for polynomial mappings $F = (F_1, \ldots, , F_n)$ of the form $F_i = X_i + H_i$ where H_i is a homogeneous polynomial or degree three in the variables $X_1, \ldots, X_n$. These are called cubic homogeneous maps. The next reduction is that one can assume that the polynomials H_i are third powers of linear forms, and prove the conjecture for any n. These reduction is based on the so called Gorni-Zampieri pairing, that allows to associate to a given cubic homogeneous map some cubic linear map, probably with more variables, and viceversa. The thesis contains an algorithm using pseudo-code to describe explicitly this procedure. A nice example is given, where the cubic homogeneous map has 4 variables but the associated cubic linear map involves 24 variables. Then, by using the previous Theorem 1.3.10, it is obtained in Theorem 2.4.1 another improvement of the Wronskian criterion for a polynomial mapping F to be invertible proved by T. Crespo and Z. Hajto in Theorem 2 of [12]. This result is then specified for cubic linear maps showing that the number of wronskians to check really decreases with this improvement. A concrete example (the famous Nagata automorphism) is then considered.

The third chapter contains the main result of the thesis. Given a polynomial mapping $F : K^n \to K^n$ and a polynomial map $P(X) \in K[X_1, \ldots, x_n]^n$ it is possible to construct inductively a sequence of polynomials maps $P_i(X)$ such that if $P_m(X) = 0$ for some positive integer m, then the polynomial map $G(X) = \sum_{i=0}^{m-1}(-1)^i P_i(X)$ satisfies that $G(F(X)) - P(X) = 0$. Hence if this happens with the identity map $P(X) = (X_1, \ldots, X_n)$ one gets that F is invertible and $F^{-1} = G$. This sequence of polynomial maps is constructed by taking successive derivatives of P for the (natural) derivation Δ_F on $K[X]^n$ defined as $\Delta_F(P) = P \circ F - P$. A critical point is that the original polynomial $P(X)$ can always be recovered in terms of the polynomials $P_i(X)$ (for a finite number of steps, even if the algorithm does not stop) with a simple combinatorial formula. In case the algorithm stops in a finite number of steps for the identity map, F is called to be Pascal finite. (The name comes from the fact that if F is Pascal finite then it has a vanishing polynomial whose coefficients come from the combinatorial Pascal triangle.) By the very definition of locally finite polynomial maps, any Pascal finite polynomial mapping is locally finite. In Proposition 3.2.2 it is shown that the converse also holds for a special kind polynomial mappings which include the cubic linear and the cubic homogeneous maps. In case that the algorithm does not stop, in Theorem 3.2.3 and for the same kind of polynomial mappings F as above it is given an equivalent condition for F to be invertible in terms of a finite number of polynomials among the ones produced by the algorithm, and a formula to compute the inverse of F also in terms of the same polynomials. This is a remarkable result because it does not require any finiteness condition and it is based in the fact already pointed out that the original polynomial $P(X)$ can always be recovered in terms of the polynomials $P_i(X)$. The rest of the chapter is devoted to implement the algorithm in SAGE, providing several interesting explicit examples. Finally, it is also estimated the computational complexity of checking if a given polynomial mapping is not Pascal finite. Chapter 4 is devoted to study in more detail Pascal finite automorphisms. Several results are then shown. Among them, that triangulable polynomial automorphisms are Pascal finite (Theorem 4.1.3) and that the property is preserved by the Gorni-Zampieri pairing (Proposition 4.3). It is remarkable the fact that the famous Nagata automorphism is Pascal finite, although it is known to be not tame.

There are three appendices. Appendix A describes shortly the main aspects of some of the most common computer algebra systems: Magma, Maple, Wolfram Mathematica and SageMath. Implementations of the main algorithm are given for each system as well as performance tests to compare them. The conclusion is that they behave quite similar, maybe SageMath takes more time but consumes less memory. Then, improvements of the algorithm are done with the advantage in each case of using less memory by one side but increasing the time on the other. Since SageMath is an open-source software this allows to see in detail how the implementation of the algorithm works, something that is done at the end of the appendix. Appendix B is devoted to use more advanced programming techniques in order to decrease the execution time and memory consumption of the implementations. This is done by using parallel programming, MPI implementations and Python language in order to extract the best options of each of the computer algebra systems. Finally, in Appendix C very large and complicated expressions that appeared in the computation of several examples in the thesis are given separately.

Presentation, language and use of the literature

The thesis is well written, with a logic structure of the chapters and sections. The general style is smooth and the arguments are usually clear, although some of them need cumbersome computations with polynomials. The introduced background is enough to read the thesis without too many queries to external sources, albeit it is convenient to check some of them to have a better idea of the scope and context of the thesis. The mathematical language and symbology is correct and clear. There are some misprints, but they do not prevent the right understanding of the statements. There is a big number of very helpful concrete examples: this is an outstanding point in the thesis. The computational tests are really well done and illustrate satisfactorily the complexity of the effective study of polynomial maps in the context of the Jacobian conjecture. One can realize that there is a lot of computational work done that do not explicitly appear in the thesis. The list of references is complete and correctly cited along the thesis.

Originality and sufficiency of the thesis

The content of the thesis fits into the study of one of the most important and attractive mathematical open problems nowadays: the Jacobian conjecture. It starts with an approach by means of a very active and reflourished research area: differential Galois theory. From that, an effective approach to the problem is done. This allows a computational treatment of the problem, also for the study of polynomial mappings from a general point of view. It recovers some known examples and provides some other meaningful new ones. The author shows a good knowledge of the needed mathematical skills and exhibits a very expertise treatment of the computational aspects. From this point of view the thesis is a good contribution to the general study of polynomials mappings. One already published paper in a good journal also guarantees the quality and the novelty of some of the results obtained in the thesis, namely Chapter 3. Two more papers are submitted for publication.

Final conclusion

As a final conclusion of all the above considerations, I think that the reviewed dissertation meets the requirements of a doctoral thesis and therefore I recommend the Faculty Council to admit it for its public defense.

In Barcelona, July 20, 2017.

Santiago Zarzuela Armengou
Full Professor of Algebra
University of Barcelona