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# Report on a thesis 'Some fully nonlinear elliptic equations in differential geometry' by Dongwei Gu

The thesis consists of two parts. Part one establishes the existence and uniqueness of a geodesic between any two points from the space of volume forms associated with a Riemannian manifold with nononegative sectional curvature. Part two is about the existence and uniqueness of solutions to the Dirichlet problem for the complex Hessian equation.

Equations studied in both parts of the thesis have their roots in differential geometry.

The geodesic equation considered in part one of the thesis appears as a generalisation of a geodesic equation obtained in the following setting:

If $(M, \omega)$ is an $2n$-dimensional compact Kähler manifold with a Kähler form $\omega$, then the infinite dimensional space of Kähler potentials

$$H = \{\phi \in C^\infty(M, \mathbb{R}) \mid \omega + i\partial\bar{\partial}\phi > 0\}$$

is naturally equipped with the Riemannian structure. Noting that the tangent space $\mathrm{T}_\phi H$ at $\phi$ to $H$ may be identified with $C^\infty(M, R)$, one defines the Riemannian metric $< \cdot, \cdot >$ on $H$, which point by point is given by:

$$< \psi_1, \psi_2 >_\phi := \frac{1}{V}\int_M \psi_1\psi_2(\omega + i\partial\bar{\partial}\phi)^n, \qquad \forall \psi_1, \psi_2 \in C^\infty(M, \mathbb{R}) \simeq \mathrm{T}_\phi H.$$

Motivated by a question about an existence of a certain special Kähler metrics, Donaldson conjectured that every two points in $H$ are connected by a geodesic in the metric $< \cdot, \cdot >$.

The main result of part one of the thesis parallels the above setting. It is concerned with the infinite dimensional space $V$ of volume forms, associated with a connected compact Riemannian manifold $(M, g)$. Such a mnaifold has its natural volume form $\mathrm{d}g$ associated with the metric, and its volume $V_0 = \int_M \mathrm{d}g$. The space $V$ of volume forms is

$$V = \{\phi \in C^\infty(M, \mathbb{R}) \mid 1 + \triangle\phi > 0\},$$

where the $\triangle$ is the Laplacian for $g$. It can be interpreted as the space of volume forms, since $\mathrm{d}^\phi g = (1 + \triangle\phi)\mathrm{d}g$ is another volume form integrating to the same volume $\int_M \mathrm{d}^\phi g =$

$\int_M \mathrm{d}g = V_0$ of the manifold $M$. Similarly to the Kähler case the space $V$ is a Riemannian manifold with the metric $< \cdot, \cdot >$ given by

$$< \psi_1, \psi_2 >_\phi := \frac{1}{V} \int_M \psi_1 \psi_2 \mathrm{d}^\phi g, \qquad \forall \psi_1, \psi_2 \in C^\infty(M, \mathbb{R}) \simeq \mathrm{T}_\phi V.$$

In particular the question about the existence and uniqueness of geodesics between any two points in $V$ can be asked again. This is the main question studied in part one of the thesis. It culminates in the following theorem:

*For any two points in the space $V$ of volume forms associated with a closed Riemannian manifold $(M, g)$ with non-negative sectional curvature there exists a unique $C^{1,1}$ geodesic segment conecting these two points.*

This is the main original result of part one of the thesis.

Interestingly, the geodesic equations for the spaces $H$ and $V$ considered in the thesis are closely related to the Dirichlet problem for the homogeneous Monge-Ampere equations. Part two of the thesis concerns with generalizations of these, namely with the folowing equations

$$\begin{aligned} (\eta + \mathrm{dd}^c u)^m \wedge \alpha^{n-m} &= f \alpha^n \\ u = \phi \quad \text{on} \quad &\partial M \end{aligned} \tag{1}$$

for a real-valued function $u \in C^\infty(\bar{M})$, where $\bar{M} = M \cup \partial M$ is a compact Hermitian manifold with a smooth boundary $\partial M$, $\eta$ is a given 1-form, $f$ a given function on $\bar{M}$, and $\phi$ is a given function on $\partial M$. The solutions to (1) are supposed to satify inequalities

$$(\eta + \mathrm{dd}^c u)^k \wedge \alpha^{n-k} > 0, \quad \forall k = 1, 2, \dots m = \mathrm{dim} M. \tag{2}$$

The main result of this part of the thesis consists in the theorem which assures an existence and uniqueness solution to (1)-(2) in small balls.

The thesis is well written and results are clearly stated.

I find the results of this thesis as more than sufficient for PhD degree in mathematics. I recommend the thesis for a public defense.

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