REVIEW ON MARIA TRYBULA'S PHD THESIS "THE BERGMAN KERNEL FUNCTION AND RELATED TOPICS"

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Chapter I: This part of the thesis deals mainly with the Bergman kernel and the Bergman metric.

Part 1.1 deals with an effective formula of the Bergman metric for the symmetrized bidisc. The proof is based on the observation by Misra, Roy, and Zhang that the Schur polynomials establish an orthonormal basis for $\mathbb{A}^2(\mathbb{G}_n)$ and the Jacobi-Trudy identities (where the conjugate partition is used here is not clear). It finally leads to calculate the operator norm of some linear operator A on an ℓ^2-sequence space. It should be emphasized that the method of proof here is not based on an intense work with the square integrable holomorphic functions on $\mathbb{G}_2$. Although the Bergman kernel is known the argument used here avoids a perhaps long calculation of the derivatives of $\log K_{\mathbb{G}_2}$. It is substituted by a clever functional analysis method.

The choice of the variables s_j seems to be not optimal since the elementary symmetric polynomials carry the same letters.

It remains unclear whether this kind of approach also leads to effective formulas in the case of $\mathbb{G}_n$, $n > 2$, or even more, whether it can be also used for other situations.

Part 1.2 presents a variation of Bell's transformation formula for proper holomorphic mappings.

Part 1.3 investigates the Lu Qi-Keng problem for the tetrablock $\mathbb{E}$. It turns out that $\mathbb{E}$ is not a Lu Qi-Keng domain but it is the proper image of a Lu Qi-Keng domain, namely the domain of all symmetric 2×2 matrices with norm less than 1. Effective zeros of $K_{\mathbb{E}}$ are given. The result is a consequence of Bell's result from the part before.

Part 1.4 contains a generalization of a result of Rudin which allows to conclude that a certain holomorphic mapping $F : D \longrightarrow \mathbb{C}^n$, $D \subset \mathbb{C}^n$, is automatically proper onto its open images. The main tool here consists of a finite group $\mathcal{U}$ of topological transformations of D for which F is precisely $\mathcal{U}$-invariant, i.e. $F(z) = F(w)$ if and only if there is a $U \in \mathcal{U}$ with $U(z) = w$. Here the situation is studied when the domain above is given by the following Hartogs domain $G := \{(z, w) \in \mathbb{C}^k \times D :$

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$|z_j| < \varphi_j(w), j = 1, \dots, k\}$, where φ_j are positive continuous functions on D, and the mapping is the following one: $G \ni (z, w) \longmapsto (z, F(w))$. Applications are given only for $k = 0$. The case $k = 1$ is mentioned without details.

The sense of such a result is to avoid the ad hoc proofs that the mapping defining the symmetrized polydisc is a proper one.

Summarizing: The parts so far deal with the Bergman metric and properties of special domains which have appeared during the study of μ-synthesis. The most interesting part is Part 1.1 because of the new method used to get the Bergman metric.

Part 1.5 is the main and most difficult part in this chapter. It deals with the Bergman distance for plane domains.

The comments after Lemma 1.5.4 that the estimate (1.5.2) is quite natural needs a stronger reasoning in case it is true.

The goal of this part is to continue investigations done by Nikolov for Dini-smooth bounded plane domains. Nikolov has given lower and upper estimates for the Carathéodory distance $c_D(z, w)$, resp. the Kobayashi distance $k_D(z, w)$, in terms of the Euclidean distance $|z - w|$ of the involved points and their boundary distances $d_D(z)$ and $d_D(w)$. The study of the Bergman distance remained open. This gap is now closed by the result stated in Proposition 1.5.8, resp. Proposition 1.5.10.

The main tools used to prove these results are a localization result for the Kobayashi-Royden metric near a strictly pseudoconvex boundary point due to Forstneric-Rosay and a deep result of Balogh-Bonk on a strictly pseudoconvex domain D in $\mathbb{C}^n$, $n \geq 2$. The last one states a lower/upper estimate for the integrated form of a pseudometric that satisfies certain boundary behavior in terms of the so-called Carnot-Carathéodory metric on ∂D and the distances to the boundary of D. In particular, the assumptions needed in the Balogh-Bonk result are fulfilled for the Kobayashi-Royden metric on a strictly pseudoconvex domain (in particular, the unit ball) and for a somehow modified Kobayashi-Royden pseudometric on the ball. Hence, the result can be applied.

The proof seems to be correct. Nevertheless, the way it is presented needs some comments. After the first reduction it is unclear whether the use of Warschawski's Theorem for a boundary point p of D gives a uniform estimate in the constants involved in $d_{\mathbb{D}}(z) \sim d_D(F(z))$; note that the construction heavily depends on p. Note also that r_0 depends on p. The first paragraph in the proof is simply superfluous; only r, r_1, r_2, and r_3 are needed. More, it seems that r is the same as r_0.

Recall that Nikolov proved that for a finitely connected plane domain (without isolated boundary points) the quotient $\frac{c_D(z,w)}{k_D(z,w)}$ tends to one if w tends to the boundary and z remains in some compact part of D (in fact, the convergence is uniform in z). A similar result is known for strongly pseudoconvex domains due to Venturini. Now, in the thesis a similar boundary behavior is given for the quotient $\frac{b_D}{c_D}$ and $\frac{b_D}{k_D}$.

Summarizing: Theorem 1.5.8 is the most important result here. And the idea to use the Balogh-Bonk result proves the author's ability to find the correct and even unusual tools overcoming the problems for the Bergman case.

Part 1.6 contains some minor contributions improving results by Costara.

Chapter II: It deals mainly with the Kobayashi pseudodistance.

Part 2.1 discusses the size of the ℓ_D, k_D, and c_D-balls for domains D, which are convex or $\mathbb{C}$-convex and do not contain any complex line. The main tool used here is the so–called minimal basis at a point of D which was introduced by McNeal in 1992. The main result may be described as follows.

Let $D \subset \mathbb{C}^n$ be a domain with no complex lines inside and fix a point $q \in D$. Assume that the standard base $e_1, \dots, e_n$ of $\mathbb{C}^n$ is minimal for D at q, i.e.

$q + d_D(q)e_1 \in \partial D$, $q + d_{D_2}(q)e_2 \in \partial D$, $q + d_{D_3}(q)e_3 \in \partial D$ etc., where $D_1 := D$, $D_2 := \{(z_2, \dots, z_n) \in \mathbb{C}^{n-1} : (q, z_2, \dots, z_n) \in D\}$, $D_3 := \{z_3, \dots, z_n) \in \mathbb{C}^{n-2} : (q_1, q_2, z_3, \cdots, z_n) \in D\}$, etc.

Write $\tau_j := d_{D_j}(q)$ and $\tau := (\tau_1, \dots, \tau_n)$ and note that $\tau_j \leq \tau_{j+1}$. Then the result reads as:

1) if D is linearly convex and if $z \in \mathbb{C}^n$ with $\max\{\frac{|z_j - q_j|}{\tau_j}\} < \frac{e^{2r}-1}{n(e^{2r}+1)}$, then $z \in D$ and $\ell_D(z, q) < r$;

2) if D is convex and if $z \in D$ with $c_D(q, z) < r$, then $\max\{\frac{|z_j - q_j|}{\tau_j}\} < e^{2r} - 1$;

3) if D is $\mathbb{C}$-convex and if $z \in D$ with $c_D(z, q) < r$, then $\max\{\frac{|z_j - q_j|}{\tau_j}\} < e^{4r} - 1$.

In particular, the Kobayashi ball $B_{k_D}(q, r)$ contains a certain polydisc and is contained in another one and the corresponding multiradii are given in terms of τ.

The proof of Theorem 2.1.3 heavily depends on Lemma 2.1.4 in which lower estimates are given for c_D in the above situation of D. Its proof uses Koebes 1/4- theorem together with a clever relation between a new ad hoc distance, its integrated form and their relations.

Besides of Lemma 2.1.4 (NOT Lemma 2.1.5 as stated in the text) a specific coordinate transform (exploiting C-convexity) and the product property leads finally to the proof of Theorem 2.1.3.

The formulation of Theorem 2.1.6 is less precise. By assumption, $U \cap D$ is weakly linearly convex, convex or $\mathbb{C}$-convex but it remains unclear which are the precise requirements for q, z. It seems that some localization result has to be used in order to understand the required assumption. Therefore, the author's statement that Theorem 2.1.6 is "merely an application of Theorem 2.1.3" is far from being obvious. In fact, such a statement violates in some sense the very positive judgement of the former extremely clear representation of Theorem 2.1.3.

Part 2.2 deals with Gromov hyperbolicity. It remains unclear why the study of this kind of hyperbolicity is important in Complex Analysis.

There are two definitions of δ-hyperbolic formulated. The first one is formulated for geodesic spaces and deal with the size of geodesic triangles. The second notion works in any metric space (X, d) and is defined via the so-called Gromov product. If $S(p, q, x, w) \leq 2\delta$ for all $p, q, x, w \in X$, X is said δ-hyperbolic (it seems that the notion δ-Gromov hyperbolicity should be the correct one in order to distinguish from the one above).

In the case that (X, d) is an intrinsic metric space, it is mentioned that both notions are the same. Why an intrinsic metric space is a geodesic metric space ? In any case it should be emphasized that for a domain $D \subset \mathbb{C}^n$ the space (D, k_D) is intrinsic or in other words, k_D is an inner distance.

Unclear is the statement of Corollary 2.2.6 and the role of Example 2.2.7.

To apply Theorem 2.2.8 the author quotes Theorem 12 from NPZ-2, in which a comparison is given between the Carathéodory-Reiffen, the Bergman, and the Kobayashi-Royden-metric. After integration one ends up with a comparison between the inner Carathéodory, the Kobayashi and the Bergman distance. Therefore, c^i_D, k_D, and b_D are bilipschitz equivalent. Why is the same true for c_D as it is needed ?

After this kind of introduction general results are given. The first one states that, in general, Gromov hyperbolicity fails for the Cartesian product $X_1 \times X_2$ with $d := \max\{d_1, d_2\}$, if both factors (X_j, d_j) have infinite distances and (X_1, d_1) is intrinsic. Or more precise, the product is Gromov hyperbolic if and only if one factor is Gromov hyperbolic and the other has a bounded distance. In particular, the bidisc is not Gromov hyperbolic with respect to the Kobayashi distance.

Moreover, it is shown that also the symmetrized polydisc is not Gromov hyperbolic for c, k and that the symmetrized bidisc is not Gromov hyperbolic for b. The idea of the proof is to study the Gromov property for certain images of points on the "diagonal" of the polydisc. Without proof it is also mentioned that the tetrablock is not Gromov hyperbolic.

The main and the most difficult part in this section deals with the question which influence the existence of an analytic disc in the boundary of a convex domain D may have with respect to Gromov hyperbolicity. Such a result was proved in case that D has a $\mathcal{C}^\infty$-boundary (see Gaussier-Seshadri). Here the author gives the following result: if $D \subset \mathbb{C}^2$ is convex, without a complex line, if the boundary is of class $\mathcal{C}^{1,1}$, and if ∂D contains an analytic disc, then D is not Gromov hyperbolic for k_D. Is remains open, what happens in case $n > 2$ (see also the Remark 2.2.16).

While Gaussier-Seshadri argue via general results from the Gromov theory, the author here gives a direct proof exploiting only the geometry of D and estimates for the Kobayashi distance. The reasoning is very tricky and absolutely non trivial. In some geometric argument more details (instead of some kind of hand-waving) would be helpful for understanding.

It should be mentioned that there are two papers by Zimmer in which very general results for $\mathbb{C}$-convex domains in this context are proved. I am wondering why the author is not quoting these two papers.

In Theorem 2.2.17 non Gromov hyperbolicity is related to some type conditions (in the sense of D'Angelo) of the boundary of D; note that the boundary is not assumed to be of class $\mathcal{C}^\infty$ but locally of class $\mathcal{C}^{1,1}$. The proof is based on very nice, non trivial geometric investigations together with estimates of the Kobayashi distance.

Simple examples may show that there are Gromov hyperbolic domains which are not pseudoconvex. Take, for example, the unit ball minus some thin subset. The author shows that there are even smooth bounded domains which are not pseudoconvex but Gromov hyperbolic. Take a strictly pseudoconvex domain G and a subdomain $D \subset\subset G$ with $\mathcal{C}^2$-boundary such that at any of its boundary points its Levi form has at least one positive eigenvalue. Then $G \setminus \overline{D}$ is not pseudoconvex, but nevertheless it is Gromov hyperbolic for k. The main reason for this positive result is that $k_G - k_{G\setminus\overline{D}}$ is bounded on $(G \setminus \overline{D})^2$. To see this, mainly a result by Krantz is used which describes the boundary behavior of the Kobayashi-Royden metric in direction of the normal vector.

A similar result remains true if D is replaced by a polycylinder.

Summarizing: this part of the thesis is highly non trivial; it contains besides a clever use of estimates for the Kobayashi distance a lot of tricky geometric ideas.

Part 2.3 contains an effective formula of the Carathéodory-Reiffen metric for the symmetrized bidisc (at points $(0,p)$, $0 < p < 1$) exploiting the more abstract formula due to Costara and Agler-Young. The proof consists of some simple calculations. Nevertheless, the formula shows how complicated concrete formulas may look like. The Remark 2.3.3 emphasizes that $\gamma_{\mathbb{G}_2}((0,p);(1,\cdot))$ is not differentiable but, of course, there are much simpler examples with this kind of phenomena, e.g. the bidisc. See also the first paragraph on page 46.

Summary: The author of this thesis proves her high mathematical ability to solve non trivial problems in the field of geometric complex analysis. The proofs definitely show that she is able to combine her intensive knowledge about the invariant functions together with original nice geometric ideas. Hence there is no doubt that the thesis satisfies all requirements to get a Ph.D. in mathematics. Nevertheless, the way the thesis is written is far from being optimal. The representation could have been improved in an essential way. So my final conclusion using the traditional notes for a PhD thesis is to evaluate this work with the note *magna cum laude.*

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