JAGIELLONIAN UNIVERSITY

Faculty of Mathematics and Computer Science

Iwo Biborski

Geometric and differential properties of closed sets definable in quasianalytic structures

PhD dissertation under the supervision of
Dr Hab. Krzysztof Nowak

Kraków 2014

I would like to thank my PhD advisor dr hab. Krzysztof Nowak for the great effort, the priceless mathematical inspiration and the time devoted to my development.

Contents

Introduction

This dissertation is concerned with a quasi-analytic structure $\mathcal{R}$, i.e. the expansion of the real field $\mathbb{R}$ by restricted Q-analytic functions. The sets definable (with parameters) in the structure $\mathcal{R}$ are precisely those subsets of $\mathbb{R}^n$ that are globally quasi-subanalytic, i.e. quasi-subanalytic in a semialgebraic compactification of $\mathbb{R}^n$ (quasi-subanalytic including infinity).

Recall precise definitions. Fix a quasi-analytic system $\mathcal{Q} = (Q_n)_{n\in\mathbb{N}}$ of sheaves of local $\mathbb{R}$-algebras of smooth functions on $\mathbb{R}^n$. For each open subset $U \subset \mathbb{R}^n$, $Q(U) = Q_n(U)$ is thus a subalgebra of the algebra $\mathcal{C}^\infty(U)$ of real smooth functions on U. By a Q-analytic function (or Q-function for short), we mean any function $f \in Q(U)$. Similarly $f = (f_1, \ldots, f_k) : U \to \mathbb{R}^k$ is called Q-analytic (or a Q-map) if so are its components $f_1, \ldots, f_k$. The following conditions on the system $\mathcal{Q}$ are imposed:

1. each algebra $\mathcal{Q}(U)$ contains the restrictions of polynomials;

2. $\mathcal{Q}$ is closed under composition, i.e. the composition of Q-mappings is a Q-mapping, whenever it is well defined;

3. $\mathcal{Q}$ is closed under inverse, i.e. if $\varphi : U \to V$ is a Q-mapping between open subsets $U, V \subset \mathbb{R}^n$, $a \in U$, $b \in V$ and if $\frac{\partial\varphi}{\partial x}(a) \neq 0$, then there are neighborhoods U_a and V_b of a and b respectively, and Q-diffeomorphism $\psi : V_b \to U_a$ such that $\varphi \circ \psi$ is the identity mapping on V_b;

4. $\mathcal{Q}$ is closed under differentation;

5. $\mathcal{Q}$ is closed under division by a coordinate, i.e. if $f \in \mathcal{Q}(U)$ and $f(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n) = 0$ as a function in the variables x_j, $j \neq i$, then $f(x) = (x_i - a_i)g(x)$ with some $g \in \mathcal{Q}$;

6. $\mathcal{Q}$ is quasianalytic, i.e if $f \in \mathcal{Q}(U)$ and the Taylor series $\hat{f}_a$ of f at $a \in U$ is 0, then f is 0 in a vicinity of a.

Q-analytic maps give rise, in the ordinary manner, to the category Q of Q-manifolds and Q-maps, which is a subcategory of that of smooth manifolds and smooth maps. Similarly, Q-analytic, Q-semianalytic and Q-subanalytic sets can be defined.

Denote by $\mathcal{R} = \mathcal{R}_Q$ the expansion of the real field $\mathbb{R}$ by restricted Q-analytic functions, i.e. functions of the form

$$\widetilde{f}(x) = \begin{cases} f(x), & \text{if } x \in [-1,1]^n \\ 0, & \text{otherwise} \end{cases}$$

where $f(x)$ is a Q-function in the vicinity of the compact cube $[-1,1]^n$. The structure $\mathcal{R} = \mathcal{R}_Q$ is model complete and o-minimal (cf. [31],[20],[21],[22]).

We shall investigate certain natural, metric, algebro-geometric and differential properties of closed sets definable in the structure $\mathcal{R}$, including:

- composite function property;
- uniform Chevalley estimate;
- Zariski semicontinuity of the diagram of initial exponents;
- semicontinuity of the Hilbert- Samuel function;
- semicohernce;
- stratification by the diagram of initial exponents;

their definitions are provided later in this chapter. Our research is inspired by the famous paper [5] by E. Bierstone and P. Milman, where the equivalence of these properties is established in the classical case of subanalytic sets. In this manner, each of those properties characterizes the class of subanalytic sets that are tame from the point of view of local analytic geometry. The basic tools applied there are:

- the uniformization theorem;
- Hironaka's diagram of initial exponents and the formal division algorithm of Grauert-Hironaka;
- the formalism of jets;
- trivialization and stratification of definable maps;
- the uniform Chevalley estimate;
- an elementary lemma from linear algebra.

In addition to the above tools, E. Bierstone and P. Milman make use of some results (as for example flatness or the noetherianity of the local analytic rings) which are unavailable in the quasianalytic settings.

The main purpose of this dissertation is to carry over the results of E. Bierstone and P. Milman to the case of closed sets definable in the structure $\mathcal{R}$. A stimulus for this work was also paper [24] by K.J. Nowak, which demonstrates that the classical Glaeser composite function theorem remains valid in quasianalytic structures.

In the first chapter we present in more detail the basic tools mentioned above, and demonstrate that they are available in the quasianalytic settings. For instance, while the classical uniformization theorem for closed subanalytic sets can be found in many papers (see [3], for example), its quasianalytic counterpart has been established in [6] only for Q-analytic sets. We shall deduce the general quasianalytic version from the theorem on decomposition into special cubes due to K.J Nowak [20]. Moreover, the last theorem enables avoiding certain other tools applied by E. Bierstone and P. Milman (as flatness), which are unavailable here. Another example are theorems on trivialization and stratification of definable maps. While it is rather doubtful whether the original proofs by Hardt can be carried over to the quasianalytic settings, the approach by Łojasiewicz ([14]), where these theorems are direct consequence of his equitriangulation of a subanalytic family, can be easily adapted to general o-minimal structures with smooth cell decomposition.

Chapter 2 is devoted to the division algorithm of Grauert-Hironaka and the generic diagram of initial exponents constructed by E. Bierstone and P. Milman in [5], recalled here for the reader's convenience.

Let us emphasize that the proofs of some implications given in [5] for the classical subanalytic case can be repeated almost verbatim in the quasianalytic case. This will be explained in Chapter 7. The remaining chapters of this dissertation are devoted to those implications whose proofs require either a different approach or, at least, a considerable modification of the approach from [5].

Let us provide necessary definitions. We denote by $\mathcal{O}_b$ the ring of germs of Q-analytic functions at b, and by $\widehat{\mathcal{O}}_b$ we denote it's completion.

Let $\varphi : M \to \mathbb{R}^n$ be a proper Q-analytic map. Let $X = \varphi(M)$. Let $b \in X$. It is clear, that for any $a \in \varphi^{-1}(b)$, φ induces the local ring homomorphisms $\varphi_a^* : \mathcal{O}_b \to \mathcal{O}_a$ and $\hat{\varphi}_a^* : \widehat{\mathcal{O}}_b \to \widehat{\mathcal{O}}_a$.

Composite differentiable function. Let M and B be the Q-analytic manifolds, i.e. the manifolds with Q-analytic charts. Let $\varphi : M \to N$ be a proper Q-analytic mapping.

Definition 0.1. Put

$$(\varphi^* C^\infty(N))\widehat{\ }:=$$
$$\left\{ f \in \mathcal{C}^\infty(M) : \forall_{b \in \varphi(M)} \exists_{G_b \in \widehat{\mathcal{O}}_b} : \left(\hat{f}_a = \hat{\varphi}_a^*(G_b)\, , \forall_{a \in \varphi^{-1}(b)} \right) \right\}.$$

We say that $f \in \mathcal{C}^\infty(M)$ is formally composed with φ if $f \in (\varphi^* \mathcal{C}^\infty(N))\widehat{\ }$.

Let $\varphi : M \to \mathbb{R}^n$ be a proper Q-analytic mapping and let $Z \subset \mathbb{R}^n$ be a closed set. We denote by $\mathcal{C}^\infty(\mathbb{R}^n; Z)$ the Frechet algebra of smooth functions from $\mathbb{R}^n$ which are flat on Z. Then φ induces a homomorphism $\varphi^* : \mathcal{C}^\infty(\mathbb{R}^n; Z) \to \mathcal{C}^\infty(M; \varphi^{-1}(Z))$. It is clear that $(\varphi^* \mathcal{C}^\infty(\mathbb{R}^n, Z))\widehat{\ } := (\varphi^* \mathcal{C}^\infty(\mathbb{R}^n))\widehat{\ } \cap \mathcal{C}^\infty(M; \varphi^{-1}(Z))$ is closed in $\mathcal{C}^\infty(M; \varphi^{-1}(Z))$.

Definition 0.2. Let $Z \subset X$ be the quasi-subanalytic subsets of $\mathbb{R}^n$. We say that (X, Z) has the composite function property if, for any proper Q-analytic mapping $\varphi : M \to \mathbb{R}^n$ such that $\varphi(M) = X$

$$\varphi^*\mathcal{C}^\infty(\mathbb{R}^n; Z) = (\varphi^*\mathcal{C}^\infty(\mathbb{R}^n; Z))\widehat{\ }.$$

Chevalley estimate. Lex X be a closed quasi-subanalytic subset of $\mathbb{R}^n$. Put

$$\mu_{X,b}(f) := \sup\{p \in \mathbb{R} : |f(x)| \leq const|x - b|^p \, , x \in X\}$$
$$\nu_{X,b}(f) := \max\{l \in \mathbb{N} : f \in \hat{m}_b^l\},$$

where $\hat{m}_b$ is the maximal ideal in $\widehat{\mathcal{O}}_b$. By Chevalley Lemma (see Chapter 3), for each $b \in X$ and $k \in \mathbb{N}$, there exists $l \in \mathbb{N}$ which satisfy the following condition:

if $f \in \widehat{\mathcal{O}_b}$ and $\mu_{X,b}(f) > l$ then $\nu_{X,b}(f) > k$.

Definition 0.3. Let $l_X(b, k)$ be the least l as above. We call $l_X(b, k)$ a Chevalley estimate.

The diagram of initial exponents. Let $X \subset \mathbb{R}^n$ be a closed quasi-subanalytic set. By the uniformization theorem (see Theorem 2.2) there exists a proper Q-analytic mapping $\varphi : M \to \mathbb{R}^n$ such that $\varphi(M) = X$. We have the following

Definition 0.4. We denote by $\mathcal{F}_b(X) \subset \widehat{\mathcal{O}}_b$ the ideal of formal relations at b, where

$$\mathcal{F}_b(X) := \bigcap_{a \in \varphi^{-1}(b)} \operatorname{Ker} \widehat{\varphi}_a^*.$$

We identify $\widehat{\mathcal{O}}_b$ with the ring of formal power series $\mathbb{R}[[y - b]]$, where $y = (y_1, \ldots, y_n)$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Then the length of α is a sum of its coordinates: $|\alpha| = \sum_{i=1}^n \alpha_i$. We consider an ordering on $\mathbb{N}^n$ defined as follows: let $\alpha, \beta \in \mathbb{N}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$. Then $\alpha > \beta$ if and only if $(|\alpha|, \alpha_1, \ldots, \alpha_n) > (|\beta|, \beta_1, \ldots, \beta_n)$ in the lexicographic order. Let $F(y) = \sum_{\beta \in \mathbb{N}^n} F_\beta (y - b)^\beta$. By the support of F we mean the set

$$\operatorname{supp} F := \{\beta \in \mathbb{N}^n : \; F_\beta \neq 0\}.$$

We denote by $\exp F$ the minimum of $\operatorname{supp} F$ in the above ordering.

Let I be an ideal in $\mathbb{R}[[y - b]]$.

Definition 0.5. We call the set

$$\mathfrak{N}(I) := \{\exp F : \; F \in I \setminus \{0\}\}$$

the diagram of initial exponents of I.

Of course $\mathfrak{N}(I) + \mathbb{N}^n = \mathfrak{N}(I)$. Thus there exists the smallest finite set $\mathcal{B} \subset \mathfrak{N}(I)$, called the set of vertices of $\mathfrak{N}(I)$, such that $\mathfrak{N}(I) = \mathcal{B} + \mathbb{N}^n$.

For the ideal I we define the Hilbert-Samuel functions in the following way

$$H_I(k) := \dim_{\mathbb{R}} \frac{\mathbb{R}[[y-b]]}{I + (y-b)^{k+1}}, \quad k \in \mathbb{N},$$

here $(y-b)$ is the maximal ideal of $\mathbb{R}^n$. By Corollary 2.2,

$$H_I(k) = \sharp\{\alpha \in \mathbb{N}^n \setminus I : |\alpha| \leq k\}.$$

Zariski semicontinuity. Let $Z \subset X$ be closed quasi-subanalytic sets in $\mathbb{R}^n$. Let Γ be a prtially-ordered set.

Definition 0.6. We say that a function $\kappa : X \setminus Z \to \Gamma$ is quasi-subanalytic Zariski semicontinuous relatively to Z, if the following two conditions hold:

(1) for every compact $K \subset X$, the set $\kappa((X \setminus Z) \cap K)$ is finite

(2) for all $\gamma \in \Gamma$, $Z_\gamma := Z \cup \{b \in X \setminus Z : \kappa(b) \geq \gamma\}$ is closed quasi-subanalytic set.

Formal semicohernce. Let $Z \subset X$ be closed quasi-subanalytic sets in $\mathbb{R}^n$. We provide a definition of formal semicoherence in the similar way as [5], Definition 1.2.

Definition 0.7. We say that X is formally semicoherent relatively to Z, if there is a quasi-subanalytic, locally finite stratification $X = \bigcup X_i$ such that Z is a sum of strata and, for each stratum X_i disjoint with Z, there is satisfied the following condition:

for each $x \in \overline{X}_i$, there is an open neighborhood U and finitely many formal power series

$$f_{ij}(\cdot, Y) = \sum_{\alpha \in \mathbb{N}^n} f_{ij,\alpha}(\cdot) Y^\alpha$$

such that $f_{ij,\alpha}$ are Q-analytic functions on $X_i \cap U$, which are quasi-subanalytic and, for each $b \in X_i \cap U$, $\mathcal{F}_b(X)$ is generated by the formal power series

$$f_{ij}(b, y-b) = \sum_{\alpha \in \mathbb{N}^n} f_{ij,\alpha}(b)(y-b)^\alpha.$$

Definition 0.8. We say that X has a stratification by the diagram of initial exponents relatively to Z, if there is a locally finite quasi-subanalytic stratification of X such that Z is a sum of strata and the diagram of initial exponents is constant on each stratum outside Z.

Our purpose is to investigate relations between the above properties established in Definitions 0.1-0.8. We shall prove the following quasianalytic version of Theorem 1.13 from [5]:

Theorem 0.1. *Let $X \supset Z$ be closed quasi-subanalytic subsets of $\mathbb{R}^n$. Then the following properties are equivalent:*

(1) (X, Z) has a composite function property.

(2) X has a uniform Chevalley estimate, i.e. for every compact $K \subset X$ there is a function $l_K : \mathbb{N} \to \mathbb{N}$ such that $l_X(b, k) \leq l_K(k)$ for all $b \in (X \setminus Z) \cap K$.

(3) there is a quasi-subanalytic stratification of X such that Z is a sum of strata and the diagram of initial exponents is constant on each stratum disjoint with Z.

(4) The diagram of initial exponents is Zariski semicontinuous, i.e. the function $b \to \mathfrak{N}_b$ is Zariski semicontinuous, where $\mathfrak{N}_b = \mathfrak{N}(\mathcal{F}_b(X))$ for $b \in X \setminus Z$.

(5) The function $b \to H_{\mathcal{F}_b(X)}$ is Zariski semicontinuous relatively to Z.

(6) X is formally semicoherent relatively to Z.

The main difficult is to prove the implications (2) $\Rightarrow$ (3), (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5), for the proofs of which we are forced to provide different approach than the one by E. Bierstone and P. Milman.

In comparison to the proofs from [5], property (4) cannot be directly drawn form property (2). To establish the semicontinuity of the diagram of initial exponents, Birstone and Milman prove simultaneosly that certain sets $Z_{\mathfrak{N}}$ (see Definition 3.2) are subanalytic and closed, if the uniform Chevalley estimate holds ([5], Proposition 8.6). Their proof cannot be directly applied in the quasianalytic settings, because it relies on that the ring of formal power series is faithfully flat over the ring of analytic function germs, which is no longer available in the quasianalytic settings. Here we are forced to follow a different, not so direct strategy. In Chapter 3, we prove that the sets $Z_{\mathfrak{N}}$ are quasi-subanalytic if the umniform Chevalley estamate holds (Proposition 3.3), and next we prove the implication (2) $\Rightarrow$ (3). To show this, we prove that the number of the diagrams of initial exponents is finite on $K \cap X$ for every compact set K. In Chapter 4, we shall prove that the sets $Z_{\mathfrak{N}}$ are closed if the uniform Chevalley estimate holds (Theorem 4.1). In the proof, we shall develop a different approach, which consists in reducing the analysis of the diagram of initial exponents to the quasi-subanalytic arcs. The foregoing two results together yield the semicontinuity of the diagram of initial exponents.

In Chapter 5, we repeat similar ideas to establish the implication (3) $\Rightarrow$ (5). We should emphasize that Bierstone-Milman's proof of this implication essentially relies on the fact that subanalytic arcs are analytic curves and their local analytic rings are noetherian. The former is no longer true in quasianalytic structures, as shown by K.J. Nowak in the example constructed in paper [29]. Also, it is rather doubtful that the local quasianalytic ring of a Q-analytic curve at a singular point is noetherian.

In Chapter 6 we present a proof of the implication (6) $\Rightarrow$ (2). Altough the proof of this implication from [5] can be carried over to quasianalytic settings without any changes, we present our own line of reasoning. Our approach relies on an explicit description of when the multi-index belongs to the diagram of initial exponents. An advantage of such an approach is that the proof uses the basic properties of the systems of linear equations and of the diagram of initial exponents.

In Chapter 7, we explain why the proofs by Bierstone-Milman ([5]) of the remaining implications, listed below

$$\begin{aligned} (3) &\Rightarrow (1), \\ (1) &\Rightarrow (2), \\ (6) &\Rightarrow (3), \\ (5) &\Rightarrow (2), \end{aligned}$$

carry over to the quasianalytic settings.

Finally in Chapter 8 we give an example of a closed, semicoherent quasi-subanalytic set which is not subanalytic.

It would be an interesting problem to find some natural classes of quasi-subanalytic sets that are semicoherent. Unlike in the classical analytic case, it seems that even Q-analytic sets may not be semicoherent. It is possible that the closures of subsets determined by analytic equations and Q-analytic strict inequalities are semicoherent. A verification of this conjecture could be the first step towards a solution to the foregoing problem, opening a new direction for further research on the subject.

Chapter 1

Necessary Tools

In this chapter we provide some necessary tools. First of all we present the concept of covering a compact quasi-subanalytic set by the special cubes due to K.J. Nowak ([20]). We use it to prove uniformization theorem in general quasi-subanalytic case. Moreover, a covering of compact quasi-subanalytic set enables us to prove Lemma 1, which is fundamental for further investigations and cannot be proved in the same way as in analytic settings.

Next we recall the formalism of jets (see [5], Chapter 4) and show some useful facts about Chevalley estimate. In particular we recall Chevalley lemma and its consequences. Later on, we present stratification and trivialization theorems for the quasianalytic settings, which are adaptations of the concepts due to Łojasiewicz presented in [14]. We end this chapter with lemma about linear equations over the noetherian local rings.

Special cubes and uniformization. Let M be a Q-manifold. Let $C \subset M$.

Definition 1.1.([20]) We call C a special cube of dimension d in M if there exists a Q-mapping φ from the vicinity of $[-1,1]^d$ into M such that the restriction of ψ to $(-1,1)^d$ is a diffeomorphism onto C.

We have the following

Theorem 1.1.([20]) *A relatively compact quasi-subanalytic subset $S \subset M$ is a finite sum of special cubes.*

As an immediate corollary, we obtain the following uniformization theorem

Theorem 1.2. *Let $F \subset \mathbb{R}^n$ be a compact quasi-subanalytic set. Then there exist a Q-manifold M and a Q-analytic proper mapping $\varphi : M \to \mathbb{R}^n$ such that $\varphi(M) = X$.*

Proof. By the definition of a definable quasi-subanalytic sets, there exists a relatively compact Q-analytic subset $E \subset \mathbb{R}^n \times \mathbb{R}^p$ such that $F = pr(E)$, where $pr : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is the canonical projection. By Theorem 1.1, $E = \bigcup_{i=1}^{s} \varphi_i([-1,1]^{n_i})$. For each cube $[-1,1]^{n_i}$ there is a Q-analytic map Π_i such that

$$\Pi_i(\mathbb{S}^{n_i}) = [-1,1]^{n_i}.$$

Therefore we can write

$$F = \bigcup_{i=1}^{s} \Phi_i(\mathbb{S}^{n_i}),$$

where $\Phi_i = pr \circ \varphi_i \circ \Pi_i$ is a Q-analytic map. We can take $M = \coprod_i^s \mathbb{S}^{n_i}$, which ends the proof. $\square$

Corollary 1.1. *Let $X \subset \mathbb{R}^n$ be a closed quasi-subanalytic set. Then there exists a proper Q-analytic mapping $\varphi : M \to \mathbb{R}$ such that $X = \varphi(M)$.*

Bierstone and Milman ([6]) proved this theorem for the case, where F is Q-analytic set. By uniformization theorem, closed quasi-subanalytic sets are precisely those, which are the images of Q-analytic manifolds by a proper Q-analytic map.

We make use of Theorem 1.1 also in order to prove the following fundamental

Lemma 1.1. *Let X be a closed, quasi-subanalytic set in $\mathbb{R}^n$ and $\varphi : M \to \mathbb{R}^n$ be a proper Q-analytic mapping such that $\varphi(M) = X$. Let $b \in X$ and $G \in \widehat{O}_b$. Put*

$$S := \{a \in \varphi^{-1}(b) : \widehat{\varphi}_a^*(G) = 0\}.$$

Then S is an open and closed subset of $\varphi^{-1}(b)$.

Proof. We rearrange the proof given by M.Birstone and P.Milman for the classical analytic case. Since we do not know whether $\widehat{\mathcal{O}}_b$ is faithfully flat over $\mathcal{O}_b$, we have to use different argument for Q-analytic settings.

We can assume that M is an open neighborhood U of 0 in $\mathbb{R}^n$, $\varphi(0) = 0$ and $b = 0$, $\varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$. Then

$$G(\varphi(x+u) - \varphi(x)) = \sum_{\beta \in \mathbb{N}^n} \frac{D^\beta G(0)}{\beta!} \left(\sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} \frac{D^\alpha \varphi(x)}{\alpha!} u^\alpha \right)^\beta = \sum_{\alpha \in \mathbb{N}^n} \frac{H_\alpha(x)}{\alpha!} u^\alpha.$$

Note that each H_α is a finite linear combination of derivatives of $\varphi_i(x)$, $i = 1, \ldots, n$, thus is a Q-analytic function in some common neighborhood of 0. If $a \in \varphi^{-1}(0)$, then $\hat{\varphi}_a^*(G) = 0$ if and only if $H_\alpha(a) = 0$ for all α. Thus S is a closed subset as an intersection of $\bigcap_{\alpha \in \mathbb{N}^m} H_\alpha^{-1}(0)$ with $\varphi^{-1}(0)$.

Observe that $G(y+v) - G(y) \in (v) \cdot \mathbb{R}[[y,v]]$, where $(v) = (v_1, \ldots, v_n)$ is an ideal generated by v_i, $i = 1, \ldots, n$. Then

$$G(\varphi(x+u) - \varphi(x)) - G(\varphi(x+u)) \in (\varphi(x)) \cdot \mathbb{R}[[x,u]],$$

where $(\varphi(x))$ is the ideal generated by $\varphi_i(x)$, $i = 1, \ldots, n$. Suppose that $a = 0 \in S$, which means $\hat{\varphi}_0^*(G) = 0$. Since $G \in \operatorname{Ker} \widehat{\varphi}_0^*$, thus $G(\varphi(x+u)) = 0$ and $G(\varphi(x+u) - \varphi(x)) \in (\varphi(x)) \cdot \mathbb{R}[[x,u]]$. Since φ is a proper Q-analytic map then $\varphi^{-1}(0)$ is a compact subset of M. By Theorem 1.1, we can represent $\varphi^{-1}(0)$ as a finite sum of special cubes:

$$\varphi^{-1}(0) = \bigcup_i S_i,$$

where, for each i, S_i is a special cube, $\psi_i : V_i \to M$ is a Q-mapping from vicinity V_i of $[-1,1]^{d_i}$ and $\psi|_{(-1,1)^{d_i}}$ is a diffeomorphism onto S_i. Therefore we can cover $\varphi^{-1}(0)$ by Q-analytic arcs, since each $[-1,1]^{d_i}$ can be covered by intervals, and via mapping ψ_i we get covering of $\varphi^{-1}(0)$ by the quasi-analytic arcs.

Since $G(\varphi(x+u) - \varphi(x)) \in (\varphi(x)) \cdot \mathbb{R}[[x,u]]$, we have

$$G(\varphi(x+u) - \varphi(x)) = \sum_{\alpha \in \mathbb{N}^n} \frac{H_\alpha(x)}{\alpha!} u^\alpha = f_1(x,u)\varphi_1(x) + \cdots + f_n(x,u)\varphi_n(x),$$

for some $f_i(x,u) \in \mathbb{R}[[x,u]]$. Therefore each H_α is of the form

$$H_\alpha(x) = \sum_{j=1}^{n} \hat{f}_j(x)\varphi_i(x),$$

for some $\hat{f}_i(x) \in \mathbb{R}[[x]]$. Let U be the neighborhood of 0 in $\mathbb{R}^m$. We cover $\varphi^{-1}(0) \cap U$ by Q-analytic arcs. Let $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^m$ be a Q-analytic arc, $\gamma \subset \varphi^{-1}(0) \cap U$. Then

$$H_\alpha(\gamma(t)) = \sum_{i=1}^{n} \hat{f}_i(\gamma(t))\varphi_i(\gamma(t)).$$

Since $\gamma \subset \varphi^{-1}(0)$, $\varphi_i(\gamma(t)) = 0$, and thus H_α vanishes along γ. Therefore H_α vanishes in $\varphi^{-1}(0) \cap U$ for all $\alpha \in \mathbb{N}^m$, and thus S is an open set in $\varphi^{-1}(0)$. $\square$

Lemma 1.1 is essential for the study of the behavior of the ideal of formal relations along a closed quasi-subanalytic set X. The latter, in turn, is one of the most important objects in the theory of quasi-subanalytic sets. It is included in the corollary below

Corollary 1.2. *Let X be a closed quasi-subanalytic subset of $\mathbb{R}^n$. Let $\varphi : M \to \mathbb{R}^n$ be a proper Q-analytic mapping such that $\varphi(M) = X$. Let $b \in X$ and let s be the number of the connected components of $\varphi^{-1}(b)$. Then*

$$\mathcal{F}_b(X) = \bigcap_{i=1}^{s} \operatorname{Ker} \varphi^*_{a_i},$$

where each a_i is in a distinct connected component of $\varphi^{-1}(b)$.

Formalism of jets. We recall the formalism of jets in the quasianalytic settings. Let N be a Q-analytic manifold. Let $b \in N$, $l \in \mathbb{N}$. We write $J^l(b)$ for $\widehat{\mathcal{O}}_{N,b}/\widehat{m}^{l+1}_{N,b}$, and for $G \in \widehat{\mathcal{O}}_{N,b}$, $J^l G(b)$ is a class of G in $J^l(b)$.

Let $\varphi : M \to N$ be a Q-analytic mapping from Q-analytic manifold M. Then for any $a \in \varphi^{-1}$, the homomorphism $\widehat{\varphi}^*_a : \widehat{\mathcal{O}}_{N,b} \to \widehat{\mathcal{O}}_{M,a}$ induces a linear transformation $J^l\varphi(a) : J^l(b) \to J^l(a)$.

Let $N = \mathbb{R}^n$ and $y = (y_1, \dots, y_n)$ be the coordinates in $\mathbb{R}^n$. We can identify $\widehat{\mathcal{O}}_{N,b}$ with the ring of formal power series $\mathbb{R}[[y-b]]$ and thus we can treat $J^l(b)$ as $\mathbb{R}^p$, where $p = \binom{n+l}{l}$, and $J^lG(b) = (D^\beta G(b))_{|\beta|\le l}$.

Put $J^l_b := J^l(b) \otimes_{\mathbb{R}} \widehat{\mathcal{O}}_{N,b} = \bigoplus_{|\beta|\le l} \mathbb{R}[[y-b]]$, and let $G(y) = \sum \frac{G_\beta}{\beta!}(y-b)^\beta$. We write $J^l_bG(y)$ for $(D^\beta G(y))_{|\beta|\le l} \in J^l_b$. Let us notice that $J^lG(b)$ is a vector of constant terms in $J^l_bG(y)$.

Consider a Q-analytic mapping $\varphi : M \to \mathbb{R}^n$. Take $a \in M$, and let $\varphi(a) = b$. If $x = (x_1, \dots, x_m)$ is a system of coordinates on M in a neighborhood of a, we identify $J^l(a)$ with $\mathbb{R}^p$, $p = \binom{m+l}{l}$. Then

$$(*) \quad J^l\varphi(a) : (G_\beta)_{|\beta|\le l} \longmapsto \left(\sum_{|\beta|\le|\alpha|} G_\beta L^\beta_\alpha(a) \right)_{|\alpha|\le l},$$

where $L^\beta_\alpha(a) = \frac{1}{\beta!}\frac{\partial^\alpha \varphi^\beta}{\partial x^\alpha}(a)$. By φ^β we mean $\varphi_1^{\beta_1} \cdot \dots \cdot \varphi_n^{\beta_n}$, where $\varphi = (\varphi_1, \dots, \varphi^n)$ and $\beta = (\beta_1, \dots, \beta_n)$.

Proof. (of formula $(*)$). Let $G \in \widehat{\mathcal{O}}_b$. We have

$$\widehat{\varphi}^*_a(G) = \sum_{\beta\in\mathbb{N}^n} \frac{G_\beta}{\beta!}((\widehat{\varphi_1})_a - b_1)^{\beta_1} \cdot \dots \cdot ((\widehat{\varphi}_n)_a - b_n)^{\beta_n},$$

where $\varphi = (\varphi_1, \dots, \varphi_n)$, and $(\widehat{\varphi}_i)_a$ is the Taylor series of φ_i at $a \in M$ for $i = 1, \dots, n$. Let $(\widehat{\varphi}_i)_a - b_i = (\widehat{\psi}_i)_a$. Thus

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}\left(\sum_{\beta\in\mathbb{N}^n} \frac{G_\beta}{\beta!}((\widehat{\psi_1})_a)^{\beta_1} \dots ((\widehat{\psi}_n)_a)^{\beta_n} \right) =$$

$$\sum_{\beta\in\mathbb{N}^n} \frac{G_\beta}{\beta!} \left(\sum_{\alpha_1+\dots+\alpha_{|\beta|}=\alpha} C_{\alpha_1,\dots,\alpha_{|\beta|}} \prod_{i_1=1}^{\beta_1} \frac{\partial^{|\alpha_{i_1}|}(\widehat{\psi_1})_a}{\partial x^{\alpha_{i_1}}} \cdots \prod_{i_n=1}^{\beta_n} \frac{\partial^{|\alpha_{i_n}|}(\widehat{\psi_n})_a}{\partial x^{\alpha_{i_n}}} \right),$$

where

$$C_{\alpha_1,\dots,\alpha_{|\beta|}} = \frac{\alpha!}{\alpha_1! \dots \alpha_{|\beta|}!}.$$

By the evaluation at a, we have

$$\left(\prod_{i_1=1}^{\beta_1} \frac{\partial^{|\alpha_1|}(\widehat{\psi_1})_a}{\partial x^{\alpha_{i_1}}} \cdots \prod_{i_n=1}^{\beta_n} \frac{\partial^{|\alpha_{i_n}|}(\widehat{\psi_n})_a}{\partial x^{\alpha_n}} \right)(a) = \prod_{i_1=1}^{\beta_1} \frac{\partial^{|\alpha_{i_1}|}\varphi_1}{\partial x^{\alpha_{i_1}}}(a) \cdots \prod_{i_n=1}^{\beta n} \frac{\partial^{|\alpha_{i_n}|}\varphi_n}{\partial x^{\alpha_{i_n}}}(a)$$

Observe that if $|\beta| > |\alpha|$, then at least one $\alpha_s = 0$, for $s = 1, \dots, |\beta|$. Therefore, if $|\beta| > |\alpha|$,

$$\prod_{i_1=1}^{\beta_1} \frac{\partial^{|\alpha_{i_1}|}\varphi_1}{\partial x^{\alpha_{i_1}}}(a) \cdots \prod_{i_n=1}^{\beta n} \frac{\partial^{|\alpha_{i_n}|}\varphi_n}{\partial x^{\alpha_{i_n}}}(a) = 0,$$

since $(\widehat{\psi}_i)_a(a) = 0$. Finally we get

$$J^l\varphi(a)(G_\beta)_{\beta\in\mathbb{N}^n} = \left(\sum_{|\beta|<|\alpha|} G_\beta L^\beta_\alpha(a)\right)_{|\alpha|\le l},$$

where

$$L^\beta_\alpha(a) = \frac{1}{\beta!}C_{\alpha_1,\dots,\alpha_{|\beta|}}\prod_{i_1=1}^{\beta_1}\frac{\partial^{|\alpha_{i_1}|}\varphi_1}{\partial x^{\alpha_{i_1}}}(a)\cdots\prod_{i_n=1}^{\beta n}\frac{\partial^{|\alpha_{i_n}|}\varphi_n}{\partial x^{\alpha_{i_n}}}(a) = \frac{1}{\beta!}\frac{\partial^{|\alpha|}\varphi^\beta}{\partial x^\alpha}(a).$$

□

By the chain rule, a homomorphism $\widehat{\varphi}^*_a : \widehat{\mathcal{O}}_b \to \widehat{\mathcal{O}}_a$ induces a homomorphism $J^l(b) \to J^l(a)$ over the ring homomorphism $\widehat{\varphi}^*_a$, and thus an $\mathbb{R}[[x-a]]$-homomorphism

$$J^l_a\varphi : J^l(b) \otimes_{\mathbb{R}} \widehat{\mathcal{O}}_a \to J^l(a) \otimes_{\mathbb{R}} \widehat{\mathcal{O}}_a,$$

where, for any $G \in \widehat{\mathcal{O}}_b$, $J^l_a\varphi((\widehat{\varphi}^*_a(D^\beta G))_{|\beta|\le l}) = (D^\alpha(\widehat{\varphi}^*_a(G)))_{|\alpha|\le l}$. Let us notice, that we identify $J^l(b)\otimes_{\mathbb{R}}\widehat{\mathcal{O}}_a$ with $\bigoplus_{|\beta|\le l}\mathbb{R}[[x-a]]$ and $J^l(a)\otimes_{\mathbb{R}}\widehat{\mathcal{O}}_a$ with $\bigoplus_{|\alpha|\le l}\mathbb{R}[[x-a]]$. By evaluating at a, $J^l_a\varphi$ induces $J^l\varphi(a) : J^l(b) \to J^l(a)$. We can identify $J^l\varphi(a)$ with a matrix, which coefficients are the entries of the Taylor series of $D^\alpha\varphi^\beta/\beta!$ at a, for $|\alpha| \le l$ and $|\beta| \le l$.

Let M be a Q-analytic manifold, $\varphi : M \to \mathbb{R}^n$ a Q-analytic map, and $s \in \mathbb{N}\setminus\{0\}$. Set

$$M^s_\varphi := \{\underline{a} = (a_1,\dots,a_s) \in M^s : \varphi(a_1) = \cdots = \varphi(a_s)\}.$$

We call M^s_φ an s-fold fibre product of M with respect to φ. Notice that M^s_φ is a closed Q-analytic subset of M^s. We write $\underline{\varphi}^s$ for a natural map $\underline{\varphi}^s : M^s_\varphi \to \mathbb{R}^n$ such that $\underline{\varphi}^s(\underline{a}) = \varphi(a_1)$. Thus $\underline{\varphi} = \underline{\varphi}^s = \varphi\circ\pi_i$, where $\pi_i : M^s_\varphi \to M$ is the canonical projection.

In Chapter 4 we shall need the following homomorphism

$$J^l_{\underline{a}}\varphi : J^l(b)\otimes_{\mathbb{R}}\widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}} \to \bigoplus_{i=1}^s J^l(a_i)\otimes_{\mathbb{R}}\widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}},$$

It is the homomorphism with s components:

$$J^l(b)\otimes\widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}} \to J^l(a_i)\otimes\widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}},\ i = 1,\dots,s,$$

each of which is obtained from the homomorphism

$$J^l_{a_i}\varphi : J^l(b)\otimes\widehat{\mathcal{O}}_{a_i} \to J^l(a_i)\otimes\widehat{\mathcal{O}}_{a_i}$$

by change of base $\widehat{\pi}^*_i : \widehat{\mathcal{O}}_{a_i} \to \widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}}$. We identify $J^l_{\underline{a}}\varphi$ with

$$J^l_{\underline{a}}\varphi : \bigoplus_{|\beta|\le l}\widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}} \to \bigoplus_{i=1}^s\bigoplus_{|\alpha|\le l}\otimes_{\mathbb{R}}\widehat{\mathcal{O}}_{M^s_\varphi,\underline{a}}.$$

Let L be the germ at $\underline{a}$ of Q-analytic subspace of M^s_φ. Then we have a homomorphism

$$J^l_{\underline{a}}\varphi : J^l(b) \otimes_{\mathbb{R}} \widehat{\mathcal{O}}_{L,\underline{a}} \to \bigoplus_{i=1}^{s} J^l(a_i) \otimes_{\mathbb{R}} \widehat{\mathcal{O}}_{L,\underline{a}}$$

By evaluation at $\underline{a}$ we obtain

$$J^l\varphi(\underline{a}) : J^l(b) \to \bigoplus_{i=1}^{s} J^l_{a_i}.$$

Chevalley estimate. We recall some notation from [5], Chapter 5. Let M be a Q-analytic manifold. Consider M^s_φ, where $\varphi : M \to \mathbb{R}^n$ is a proper Q-analytic mapping and $s \in \mathbb{N}$. For $\underline{a} = (a_1, \ldots, a_s) \in M^s_\varphi$ we define(see [5], p. 748) an ideal

$$\mathcal{R}_{\underline{a}} := \bigcap_{i=1}^{s} \mathcal{R}_{a_i} = \bigcap_{i=1}^{s} \operatorname{Ker} \widehat{\varphi}^*_{a_i}.$$

For $k \in \mathbb{N}$, put

$$\mathcal{R}^k(\underline{a}) := \frac{\mathcal{R}_{\underline{a}} + \underline{\widehat{m}}^{k+1}_{\varphi(\underline{a})}}{\widehat{m}^{k+1}_{\underline{\varphi(a)}}}.$$

Let $b \in \mathbb{R}^n$ and $\Pi^k(b) : \widehat{\mathcal{O}}_b \to J^k(b)$ be the canonical projection. If $l \in \mathbb{N}$ and $l \geq k$, then we write $\Pi^{lk}(b) : J^l(b) \to J^k(b)$ for the canonical projection of jets. For the linear transformation $J^l\varphi(\underline{a}) : J^l(\underline{\varphi}(\underline{a})) \to \bigoplus_{i=1}^{s} J^l(a_i)$ we write

$$E^l := \operatorname{Ker} J^l\varphi(\underline{a}),$$
$$E^{lk} := \Pi^{lk}(\underline{\varphi}(\underline{a}))E^l(\underline{a}).$$

We have the following

Lemma 1.2.([5], Lemma 5.2) *Let $\underline{a} = (a_1, \ldots, a_s) \in M^s_\varphi$. For all $k \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that $\mathcal{R}^k(\underline{a}) = E^{lk}(\underline{a})$, or equivalently such that if $G \in \widehat{\mathcal{O}}_{\underline{\varphi(a)}}$ and $\widehat{\varphi}^*_{a_i}(G) \in \widehat{m}^{l+1}_{a_i}$ for $i = 1, \ldots, s$, then $G \in \mathcal{R}_{\underline{a}} + \widehat{m}^{k+1}_{\underline{\varphi(a)}}$.*

Let $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Let X be a closed quasi-subanalytic set in $\mathbb{R}^n$.

Definition 1.2. Let $F \in \widehat{\mathcal{O}}_b$. We define

$$\mu_{X,b}(F) := \sup\{p \in \mathbb{R} : |T^l_b F(y)| \leq \text{const}\, |y-b|^p,$$
$$y \in X,\ l = \min\{k \in \mathbb{N}, k \geq p\}\}$$
$$\nu_{X,b}(F) := \max\{l \in \mathbb{N} : F \in \widehat{m}^l_b + \mathcal{F}_b(X)\}.$$

Remark 1.1.([5], Remark 6.3) It is true that $\mu_{X,b}(F) \leq \nu_{X,b}(F)$.

Definition 1.3. Let $\varphi : M \to \mathbb{R}^n$ be a proper Q-analytic mapping, $\varphi(M) = X$. Put

$$l_X(b,k) := \min\{l \in \mathbb{N} : (F \in \widehat{\mathcal{O}}_b,\ \mu_{X,b}(F) > l) \Rightarrow \nu_{X,b}(F) > k\},$$
$$l_{\varphi^*}(b,k) := \min\{l \in \mathbb{N} : [F \in \widehat{\mathcal{O}}_b, \forall_{a\in\varphi^{-1}(b)} \nu_{M,a}(\widehat{\varphi}^*_a(F)) > l] \Rightarrow \nu_{X,b}(F) > k\}.$$

If $\underline{a} = (a_1, \dots, a_s)$, $a_i \in \varphi^{-1}(b)$, then we define

$$l_{\varphi^*}(\underline{a},k) := \min\{l \in \mathbb{N} : [F \in \widehat{\mathcal{O}}_b, \forall_{i=1\dots s}\, \nu_{M,a_i}(\widehat{\varphi}^*_{a_i}(F)) > l] \Rightarrow$$
$$F \in \bigcap_{i=1}^{s} \operatorname{Ker}\widehat{\varphi}^*_{a_i} + \widehat{m}_b^{k+1}\}.$$

Remark 1.2. By Chevalley Lemma, $l_{\varphi^*}(\underline{a},k) < \infty$. If $\underline{a}$ is an s-tuple of elements a_i such that each a_i lies in a distinct connected component of $\varphi^{-1}(b)$, thus $\operatorname{Ker}\widehat{\varphi}^*_{a_i} = \mathcal{F}_b(X)$, and $l_{\varphi^*}(b,k) \le l_{\varphi^*}(\underline{a},k)$.

The lemma below is a quasianalytic version of Lemma 6.5 from [5].

Lemma 1.3. *Let $\varphi : M \to \mathbb{R}^n$ be a proper Q-analytic mapping such that $\varphi(M) = X$. Then we have:*
(1) $l_X(b,\cdot) \le l_{\varphi^*}(b,\cdot)$,
(2) *for every compact $K \subset X$, there exists $r \ge 1$, such that $l_{\varphi^*}(b,\cdot) \le r l_X(b,\cdot), b \in K$.*

We can repeat the proof of Lemma 6.5 from [5], because we have the following quasianalytic version of Łojasiewicz inequality with parameter from [19]:

Lemma 1.4.(Łojasiewicz inequality with parameter,[19]) *Let $f, g : A \to \mathbb{R}$ be the functions defined on a definable set $A \subset \mathbb{R}^n_u \times \mathbb{R}^m_x$. Assume that $A_u := \{x \in \mathbb{R}^n : (u,x) \in A\}$ is a compact set for each $u \in \mathbb{R}^n$ and*

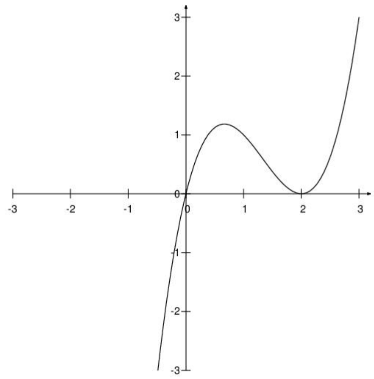
$$f_u, g_u : A_u \to \mathbb{R}^n,\ \ f_u(x) := f(u,x),\ \ g_u(x) := g(u,x),$$

are continuous. If $\{f = 0\} \subset \{g = 0\}$, then there exist an exponent $\lambda > 0$ and definable function $c : \mathbb{R}^n \to (0,\infty)$ such that

$$|f(u,x)| \ge c(u)|g(u,x)|^\lambda,\ \ (u,x) \in A$$

for some $i = 1, \dots, k$.

Remark 1.3. We should explain why the classical version of Łojasiewicz inequality is not sufficient. Indeed, the classical Łojasiewicz inequality holds for the continuous definable functions. On the other hand, a function $f(x,b) = \operatorname{dist}(x, \varphi^{-1}(b))$, which plays an important role in the proof of Lemma 1.3, need not be continuous. It is clear that if b is fixed, then $f(\cdot,b)$ is continuous, however if we fix x, $f(x,\cdot)$ is not always continuous. For example, let us consider a function $\varphi(x) = x(x-2)^2$ on a closed interval $[-3,3]$ (see the picture below).



It is clear that $\varphi(0) = \varphi(2) = 0$. Take $x = 2$. Observe, that for $b < 0$, $\varphi^{-1(b)} = \{a\}$, $a < 0$. Whence $\text{dist}(2, \varphi^{-1}(b)) > 2$ for $b < 0$. But if we put $b = 0$, then $\text{dist}(2, \varphi^{-1}(0)) = 0$. Therefore $\text{dist}(x, \varphi^{-1}(b))$ is not continuous.

Stratification and trivialization. Here we present the approach to the trivialization and stratification theorems given by Łojasiewicz ([14]). We explain how the proofs by Łojasiewicz, unlike the original ones by Hardt ([13]), can be adapted to the quasianalytic settings. Finally, we draw several corollaries, which for the classical analytic case were formulated without proofs in [5].

Let M, N be a Q-analytic manifolds. Let L be a quasi-subanalytic leaf in N. Consider a quasi-subanlytic set $E \subset L \times M$. We write $E_U := E \cap (U \times M)$, for $U \subset L$.

Definition 1.4. Let $\Pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the canonical projection,

$$\Pi(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$$

We say that the set E satisfy f-condition if, for any $u \in \mathbb{R}^{n-1}$, $\Pi^{-1}(u) \cap E$ is a finite set.

Definition 1.5. By the quasi-subanalytic equitriangulation of E over L we mean a couple $(H, \mathcal{K})$, where $\mathcal{K}$ is a simplicial complex in $\mathbb{R}^n$, $H : L \times |\mathcal{K}| \to E$ is a quasi-subanalytic homeomorphism of the form $H(t, x) = (t, h_t(x))$, such that $H(L \times \Delta)$ is a quasi-subanalytic leaf and a restriction of H to $L \times \Delta$ is a Q-analytic isomorphism on it's image, for any $\Delta \in \mathcal{K}$. In particular $(h_t, \mathcal{K})$ is a quasi-subanalytic triangulation of the fiber $E_t := \{x : (t, x) \in E\}$.

Definition 1.6. We say that equitriangulation $(H, \mathcal{K})$ is compatible with family $\mathcal{F}$ of subsets of $N \times M$ if a family of prisms of $(H, \mathcal{K})$ is compatible with $\mathcal{F}$.

Now we recall three lemmas from the paper of Łojasiewicz.

Lemma 1.5. *Let $E \subset \mathbb{R}^n$ be nowhere dense, bounded quasi-subanalytic set. Then the set of lines $\lambda \in \mathbb{P}^{n-1}$ such that $\sharp(E \cap (a + \lambda)) < \infty$ for all $a \in \mathbb{R}^n$ is dense in $\mathbb{P}^{n-1}$.*

Lemma 1.5 holds in every o-minimal structure ([11]). We need also a good direction lemma with parameter

Lemma 1.6.([11], Lemma 1.4) *Let $E \subset \mathbb{R}^{n+m}$ be a definable set in o-minimal structure, $\dim E < m+n$, $n > 0$. Then there exists a unit vector $u \in S^{n-1}$ such that*

$$\dim\{a \in \mathbb{R}^m : \sharp(\alpha \cdot u + a) \cap E = \infty\} < m.$$

Lemma 1.7. *Let $E \subset \mathbb{R}^n$ be a compact, quasi-subanalytic set satisfying f-condition. Then there exists a closed quasi-subanalytic set $F \supset E$ satisfying f-condition such that the canonical projection $\Pi_F : F \to \mathbb{R}^{n-1}$ is an open map.*

To prove the above lemma, we can repeat verbatim the original proof of Łojasiewicz([14]).

Lemma 1.8. *Let $E \subset \mathbb{R}^n$ be quasi-subanalytic, bounded set satisfying the f-condition. Then there exists a finite partition $\mathcal{L}$ into quasi-subanalytic leaves, such that for each $L \in \mathcal{L}$, $\Pi_L : L \to \Pi(L)$ is Q-analytic isomorphism.*

Proof. Let $K = \max\{\sharp(\Pi^{-1}(x)),\ x \in \Pi(E)\}$. Since E is bounded and satisfy the f-condition, $K < \infty$. Let

$$S_k := \{a \in \mathbb{R}^{n-1} : \exists_{(x_1,\ldots,x_k)\in E^k} : x_i \neq x_j,\ i \neq j,\ \Pi(x_i) = a\},\ \text{ for } k = 1,\ldots,K.$$

It is clear that each S_k is a quasi-subanalytic set and $S_1 \supset S_2 \supset \cdots \supset S_K$. Now let

$$T_k = S_k \setminus S_{k+1},\ \text{ for } k = 1,\ldots,K-1 \text{ and } T_K = S_K,\ \text{ for } k = K.$$

Each T_k is the set of those $a \in \Pi(E)$ such that $\sharp\Pi^{-1}(a) = k$ and it is quasi-subanalytic. It is enough to prove Lemma 1.8 for T_k. Since T_k is quasi-subanalytic, it has a finite number of connected components. It is sufficient to prove the conclusion of the lemma for each connected component of T_k.

Let $k \in \{1,\ldots,K\}$. Let T be a connected component of T_k. Then $\Pi|_T$ is a definable homeomorphism of T onto $\Pi(T)$. By the smooth cell decomposition, there is a finite partition of T into smooth quasi-subanalytic leaves, compatible with $\mathbb{R}^n \setminus T$. Then, on each leaf $S \subset T$, $(\Pi|_{\Pi^{-1}(S)}$ is a Q-analytic diffeomorphism. This ends the proof. □

This lemma is true in o-minimal structures with smooth cell decomposition and thus it is true in quasi-subanalytic case.

Let $\Pi_1 : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^p$ and $\Pi_2 : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^n$ be the canonical projections. We have the following

Theorem 1.3.*(Equitriangulation) Let $E_1,\ldots,E_r$ be bounded quasi-subanalytic sets in $\mathbb{R}^p \times \mathbb{R}^n$. There exists a finite partition $\mathcal{F}$ of $\Pi_1(\bigcup E_i)$ such that for each leaf T there is an equitriangulation $(H : T \times Q \to T \times Q, \mathcal{K})$ with parallelotope $Q \supset \bigcup \Pi_2(E_i)$, which is compatible with sets $E_1,\ldots,E_r$.*

The proof of Theorem 1.3 is identical as the proof for subanalytic sets in ([14]) and it is based on Lemmas 1.5, 1.6, 1.7 and 1.8. The conclusion of Theorem 1.3 leads to the following

Corollary 1.3. *The homeomorphism H provides simultaneous trivialization of E_i over T:*

$$H^{-1}((E_i)_T) := T \times F_i,$$

where F_i is a sum of some simplexes from $\mathcal{K}$. In particular we have the following theorem

Theorem 1.4.*(Trivialization) Let $f : E \to \mathbb{R}^n$ be a quasi-subanalytic, continuous map with bounded graph. There exists partition $\mathcal{T}$ of $f(E)$ into quasi-subanalytic leaves, such that for any $T \in \mathcal{T}$ there is a quasi-subanalytic homeomorphism $h : f^{-1}(T) \to F \times T$, F–quasi-subanalytic subset of $\mathbb{R}^n$, such that the following diagram is commutative:*

$$\begin{tikzcd} f^{-1}(T) \arrow[r, "h"] \arrow[dr, "f"'] & T \times F \arrow[d, "\pi"] \\ & T \end{tikzcd}$$

where π is the canonical projection.

Proof. We apply Corollary 1.3 to the graph of f, and as h we take a composition of $H^{-1}|_{f \cap (E \times T)}$ with the inverse of natural projection $\pi : f \cap (E \times T) \to f^{-1}(T)$. □

Let M and N be Q-analytic manifolds. Let $\Pi : M \times N \to N$.

Lemma 1.9. *Let $E \subset M \times N$ be a quasi-subanalytic set which is relatively compact in order to N and, for each $t \in M$, E_t is finite. Then there exists a locally finite partition Γ of E into quasi-subanalytic leaves such that, for any $T \in \Gamma$, $\Pi(T)$ is a quasi-subanalytic leaf and $\Pi|_T$ is a Q-analytic isomorphism.*

Proof. Assertion of Lemma 1.4 is a consequence of the reasoning used in the proof of Lemma 1.3 ([14]). □

Definition 1.7. We say that Q-analytic map $f : M \to N$ is a trivial submersion if there exists Q-analytic isomorphism $h : M \to N \times \Delta$ for some simplex Δ, such that the following diagram is commutative:

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[dr, "f"'] & N \times \Delta \arrow[d, "\pi"] \\ & N \end{tikzcd}$$

where π is the canonical projection.

Theorem 1.5. *(Stratification of a definable mappings.) Let $f : E \to N$ be quasi-subanalytic, continuous and proper mapping over a closed quasi-subanalytic set $E \subset M$. Let $\mathcal{F}$ and $\mathcal{G}$ be the locally finite families in M and*

N respectively. Then there exists a quasi-subanalytic stratifications $\mathcal{A}$ and $\mathcal{B}$ of M and N, compatible with $\mathcal{F}$ and $\mathcal{G}$, such that for each $A \in \mathcal{A}$ such that $A \subset E$, $f(A) \in \mathcal{B}$ and $f|_A : A \to f(A)$ is a trivial submersion.

Theorem 1.4 and 1.5 lead to the quasianalytic version of Theorem 7.1 from [5]:

Corollary 1.4.*(Stratified trivialization of quasi-subanalytic mappings.) Let E be a closed quasi-subanalytic subset of Q-analytic manifold M. Let $\varphi : E \to \mathbb{R}^n$ be a proper, continuous quasi-subanalytic mapping and let $X = \varphi(E)$. Let $\mathcal{F}$ and $\mathcal{G}$ be finite families of E and X respectively. Then there exist quasi-subanalytic stratifications $\mathcal{S}$ and $\mathcal{T}$ of E and X, such that*

(1) For each $S \in \mathcal{S}$, $\varphi(S) \in \mathcal{T}$,

(2) For each $T \in \mathcal{T}$ and $b \in T$, there is a quasi-subanalytic stratification $\mathcal{P}$ of $\varphi^{-1}(b)$ and a quasi-subanalytic homeomorphism h such that te following diagram is commutative:

$$\begin{tikzcd} \varphi^{-1}(T) \arrow[r, "h"] \arrow[dr, "\varphi"'] & T \times \varphi^{-1}(b) \arrow[d, "\pi"] \\ & T \end{tikzcd}$$

and for each $S \subset \varphi^{-1}(T)$, $h|_S$ is a Q-analytic isomorphism onto $T \times P$ for some $P \in \mathcal{P}$,

(3) $\mathcal{S}$ is compatible with $\mathcal{F}$ and $\mathcal{T}$ is compatible with $\mathcal{G}$.

Corollary 1.5. *If φ is stratified as in Corollary 1.4, then, for each stratum T, the number of connected components of the fibre $\varphi^{-1}(b)$, where $b \in T$, is constant on T.*

Corollary 1.4 is an immediate consequence of Corollary 1.3.

Let us consider a closed quasi-subanalytic set $E \subset M$ and a continuous quasi-subanalytic map $\varphi : E \to \mathbb{R}^n$. Then, for $s \in \mathbb{N}$, the subset E^s_φ is a closed quasi-subanalytic subset of s-fold fibre product M^s and we can consider a quasi-subanalytic mapping $\underline{\varphi} : E^s_\varphi \to \mathbb{R}^n$. We have the following

Corollary 1.6. *Let $(\mathcal{S}, \mathcal{T})$ be a stratification of φ as in Corollary 1.4. Take $T \in \mathcal{T}$ and denote as $\mathcal{S}^s_T$ the family of all nonempty sets of the form $(S_1 \times \cdots \times S_s) \cap E^s_\varphi$, for $S_i \in \mathcal{S}$ such that $S_i \subset \varphi^{-1}(T)$. Then $\mathcal{S}^s_T$ is a quasi-subanalytic stratification of $\underline{\varphi}^{-1}(T)$ such that each $\underline{S} \in \mathcal{S}^s_\varphi$ admits a Q-analytic isomorphism $h : \underline{S} \to T \times P$ commuting with projection on T, where P is a bounded quasi-subanalytic leaf in M^s.*

Proof. First we prove that $(S_1 \times S_s) \cap E^s_\varphi$ are quasi-subanalytic leaves in M^s. Let us fix $(S_1 \times \cdots \times S_s) \cap E^s_\varphi$. By Corollary 1.3, there exist an analytic isomorphisms $h_i : S_i \to T \times P_i$ such that $\varphi|_{S_i} = h_i \circ p$, where P_i is a stratum of $\varphi^{-1}(b)$, for some $b \in T$ and p is the projection on T.

Let us consider an isomorphism $H : S_1 \times \cdots \times S_s \to T^s \times \prod_{i=1}^s P_i$, $H(x_1, \ldots, x_s) = (h_1, \ldots, h_s)$. It is clear that $(\varphi \times \cdots \times \varphi)|_{S_1 \times \cdots \times S_s} = H \circ \Pi$, where Π is the projection of $T^s \times \prod_{i=1}^s P_i$ on T^s. Since T_s is a quasi-subanalytic leaf thus a set $\Delta = \{(a_1, \dots, a_s) \in T^s : a_1 = a_2 = \cdots = a_s\}$ is a quasi-subanalytic leaf closed in T^s and diffeomorphic to T. Whence $\Delta_1 := \Pi^{-1}(\Delta) = \Delta \times \prod_{i=1}^{s} P_i$ is a quasi-subanalytic leaf. Since H is an isomorphism, $H^{-1}(\Delta_1)$ is also a quasi-subanalytic leaf in M^s and $H^{-1}(\Delta_1) = (S_1 \times \cdots \times S_s) \cap E^s_\varphi$.

Since E^s_φ is closed then

$$\overline{(S_1 \times \cdots \times S_s) \cap E^s_\varphi} = (\overline{S_1} \times \cdots \times \overline{S_S}) \cap E^s_\varphi,$$

and

$$\overline{(S_1 \times \cdots \times S_s) \cap E^s_\varphi} \setminus (S_1 \times \cdots \times S_s) \cap E^s_\varphi = \bigcup_{i=1}^{s} (\overline{S_1} \times \cdots \times (\overline{S_i} \setminus S_i) \times \cdots \times \overline{S_S}) \cap E^s_\varphi.$$

By the assumption, $\overline{S_i} \setminus S_i$ and $\overline{S_i}$ are the sums of strata and thus a family $\mathcal{S}^s_T$ is a stratification of $\underline{\varphi}^{-1}(T)$. $\square$

Let $\underline{E}^s_\varphi$ denote a set of those $x = (x_1, \dots, x_s) \in E^s_\varphi$ such that each x_i is in a distinct connected component of $\underline{\varphi}^{-1}(\underline{\varphi}(\underline{x}))$. From Corollaries 1.5 and 1.6, we immediately obtain the following

Corollary 1.7. *Let $\varphi : E \to \mathbb{R}^n$ be a proper quasi-subanalytic map defined on a closed quasi-subanalytic set $E \subset M$. Then $\underline{E}^s_\varphi$ is a quasi-subanalytic subset of M^s.*

Linear equations over noetherian local rings. Let $\mathbb{R}[[y]]$ be a ring of the formal power series, $y = (y_1, \dots, y_n)$ and let (A, m) be a local ring such that $\widehat{A} = \mathbb{R}[[y]]$ and $\mathbb{R}[[y]]$ is faithfully flat over A. Let Φ be a matrix with coefficients in A. Let q be the number of rows of Φ. We have the following

Lemma 1.10. *Let $\widehat{\xi} \in (\mathbb{R}[[y]])^q$ and $\Phi \cdot \widehat{\xi} = 0$. Then there exists $\xi \in A^q$ such that $\Psi \cdot \xi = 0$ and $\widehat{\xi} - \xi \in \widehat{m}$.*

Proof. Let $\Phi_1, \dots, \Phi_q$ be the rows of Φ. Since $\Phi \cdot \widehat{\xi} = 0$ thus $\Phi_i \cdot \widehat{\xi} = 0$ for each $i = 1, \dots, q$. There exist $\widehat{a} \in (\widehat{m})^q$, where $\widehat{m}$ is a maximal ideal (y) in $\mathbb{R}[[y]]$, and $u \in \mathbb{R}^q$ such that $\Phi_i \cdot \widehat{a} = -\Phi_i \cdot u$. By the faithful flatness of $\mathbb{R}[[y]]$ over A we have (see [1], Commutative Algebra, Ch. I,§3,5 , prop. 10 (ii)):

$$\begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_q \end{bmatrix} \cdot \widehat{m}^p \cap A^q = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_q \end{bmatrix} \cdot m^p.$$

Thus there exists $a \in (m)^q$ such that $\Phi_i \cdot a = -\Phi_i \cdot u$. Put $\xi = a + u \in A^q$. Therefore $\Phi \cdot \xi = 0$ as desired. $\square$

Chapter 2

Generic diagram of initial exponents.

In this chapter we recall for the reader's convenience the division algorithm of Grauert–Hironaka and the construction of the generic diagram of initial exponents from [5], which carries over verbatim to the quasianalytic settings.

Division algorithm in the ring of formal power series. Let $\mathbb{R}[[y]]$ be the ring of formal power series, where $y = (y_1, \dots, y_n)$. We have the following

Theorem 2.1. *(Hironaka's division algorithm, [5], Theorem 3.1.) Let $F_1, \dots, F_s \in \mathbb{R}[[y]] \setminus \{0\}$ and $\alpha^i = exp\, F_i$, $i = 1, \dots, s$. Let $\Delta_i := (\alpha^i + \mathbb{N}^n) \setminus \bigcup_{j=1}^{i} \Delta_j$ and $\Delta := \mathbb{N}^n \setminus \bigcup_{j=1}^{s} \Delta_s$. For any $G \in \mathbb{R}[[y]]$, there exist unique Q_i, $i = 1, \dots, s$, $R \in \mathbb{R}[[y]]$ such that $G = \sum_{i=1}^{s} F_i Q_i + R$, $supp\, Q_i \in \Delta_i$ and $supp\, R \in \Delta$. Moreover, $exp\, R \geq exp\, G$, $\alpha^i + exp\, Q_i \geq expG$.*

Corollary 2.1. *([5], Corollary 3.2) Let $\mathcal{I}$ be an ideal in $\mathbb{R}[[y]]$ and $\alpha^1, \dots, \alpha^s$ be the vertices of $\mathfrak{N}(\mathcal{I})$. Let $F_1, \dots, F_s \in \mathcal{I}$ such that $exp\, F_i = \alpha^i$. Let Δ_i and Δ be the sets as in last theorem. Then*

$$(1)\ \mathfrak{N}(\mathcal{I}) = \bigcup_{i=1}^{s} \Delta_i,$$

(2) There is a unique set of generators $G_1, \dots, G_s \in \mathcal{I}$ such that

$$\mathrm{supp}\,(G_i - y^{\alpha^i}) \subset \Delta_i.$$

The system $G_1, \dots, G_s$ is called the standard basis of I. From Corollary 2.1, $\mathbb{R}[[y]] = I \oplus \mathbb{R}^{\mathfrak{N}}$, where $\mathfrak{N}$ is a diagram of initial exponents of I and $\mathbb{R}[[y]]^{\mathfrak{N}} := \{F \in \mathbb{R}[[y]] : \mathrm{supp}\, F \subset \mathbb{N}^n \setminus \mathfrak{N}\}$.

We also recall a corollary which indicates the connection between the diagram of initial exponents and the Hilbert-Samuel function. We have the following

Corollary 2.2. *([5], Corollary 3.2) Let H_I be a Hilbert-Samuel function of $\mathbb{R}[[y]]/I$. Then, for every $k \in \mathbb{N}$*

$$H_I(k) := \sharp\{\beta \in \mathbb{N}^n : \beta \notin \mathfrak{N}(I)\, and\, |\beta| \leq k\}.$$

It follows that $H_I(k)$ coincides with polynomial in k for sufficiently large k.

Lemma from linear algebra. We recall here an useful lemma from [5], Chapter 2. Let V and W be the modules over a commutative ring R.

Definition 2.1. Let $B \in \mathrm{Hom}_R(V, W)$ and $r \in \mathbb{N}$. We define

$$\mathrm{ad}^r B \in \mathrm{Hom}_R\left(W, \mathrm{Hom}_R(\bigwedge^r V, \bigwedge^{r+1} W)\right)$$

by formula

$$(\mathrm{ad}^r B)(\omega)(\eta_1 \wedge \cdots \wedge \eta_r) = \omega \wedge B\eta_1 \wedge \cdots \wedge B\eta_r,$$

where $\omega \in W$ and $\eta_1, \ldots, \eta_r \in V$.

Remark 2.1. It is clear that if $r > \mathrm{rank}\, B$, then $\mathrm{ad}^r B = 0$ and if $r = \mathrm{rank}\, B$ then $\mathrm{ad}^r B \cdot B = 0$.

We recall the following:

Lemma 2.1.*([5], Lemma 2.1) Let V and W be finite-dimensional vector spaces over a field $\mathbb{K}$. Let $B : V \to W$ be a linear transformation. Let $r := rank B$. Then*

$$\mathrm{Im}\, B = \mathrm{Ker}\, \mathrm{ad}^r B.$$

If A is a linear transformation with target W, then $A\xi + B\eta = 0$ if and only if $\xi \in \mathrm{Ker}\, \mathrm{ad}^r B \cdot A$.

The generic diagram of initial exponents. In order to introduce generic diagram of initial exponents and generic Hilbert-Samuel function, we formulate the following

Lemma 2.2. *Let A be a matrix of dimension $k \times n$ and B be a matrix of dimension $k \times m$. Consider a block matrix (A, B). Let π be a projection from $\mathbb{R}^{n+m}$ onto $\mathbb{R}^n$. Then $\pi(Ker(A, B)) = \{x \in \mathbb{R}^n : Ax \in Im B\}$.*

Proof. Let $x \in \mathbb{R}^n$. Then $x \in \pi(\mathrm{Ker}\,(A, B))$ if and only if, there exists $y \in \mathbb{R}^m$ such that $(x, y) \in \mathrm{Ker}\,(A, B)$. Hence $Ax + By = 0$, and thus $Ax = -By$. Therefore

$$x \in \pi(\mathrm{Ker}\,(A, B)) \Leftrightarrow \exists y \in \mathbb{R}^m : Ax = -By \Leftrightarrow Ax \in \mathrm{Im}\, B.$$

□

Let $L \subset M^q_\varphi$ be a Q-analytic leaf. Let $\xi = (\xi_\beta)_{|\beta| \le l}$. We can write it as $\xi = (\xi^k, \zeta^{lk})$, where $\xi^k := (\xi_\beta)_{|\beta| \le k}$ and $\zeta^{lk} := (\xi_\beta)_{k < |\beta| \le l}$. According to this decomposition we can write $J^l\varphi(\underline{a}$ as a block matrix

$$J^l\varphi(\underline{a}) = (S^{lk}(\underline{a}), T^{lk}(\underline{a})) = \begin{bmatrix} J^k(\underline{a}) & 0 \\ \star & \star \end{bmatrix},$$

for $\underline{a} \in L$ (see Chapter 1, **Formalism of jets**). By Lemma 2.2, we have

$$E^{lk}(\underline{a}) = \{\xi^k = (\xi_\beta)_{|\beta| \le k} : S^{lk}(\underline{a}) \cdot \xi^k \in \mathrm{Im}\, T^{lk}(\underline{a})\}.$$

Therefore $E^{lk}(\underline{a}) = \operatorname{Ker}\Theta^{lk}(\underline{a})$, $d^{lk} = \operatorname{rk}\Theta^{lk}(\underline{a})$, where

$$\Theta^{lk}(\underline{a}) := \operatorname{ad}^{r^{lk}} T^{lk}(\underline{a}) \cdot S^{lk}(\underline{a})$$
$$r^{lk}(\underline{a}) := \operatorname{rk} T^{lk}(\underline{a}).$$

Summing up, we obtain the following

Lemma 2.3 *Let $k \in \mathbb{N}$. For sufficiently large $l \in \mathbb{N}$, $dim\, E^{lk}(\underline{a}) = dim(\mathcal{R}^k(\underline{a}) + \widehat{m}_{\varphi(\underline{a})})/\widehat{m}_{\varphi(\underline{a})} = dim\, Ker\, ad^r T^{lk}(\underline{a}) \cdot S^{lk}(\underline{a})$.*

Proof. The conclusion of Lemma 2.3 is the consequence of the reasoning above and Lemma 1.2(Chevalley Lemma). □

Remark 2.1 The above lemma will play a crucial role in the Chapter 5 in proof that the stratification by the diagram of initial exponents implies the semicontinuity of the Hilbert-Samuel function.

Let L be a quasi-subanalytic leaf in M^s_φ. We define

$$r^{lk}(L) := \max_{\underline{a}\in L} r^{lk}(\underline{a}),$$
$$\Theta^{lk}_L(\underline{a}) := \operatorname{ad}^{r^{lk}(L)} T^{lk}(\underline{a}) S^{lk}(\underline{a}),$$
$$d^{lk}_L := \operatorname{rk}\Theta^{lk}_L(\underline{a}).$$

Put

$$d^{lk}(L) := \max_{\underline{a}} d^{lk}_L(\underline{a}).$$

Consider a set

$$Y^{lk} := \{\underline{a} \in L : \, r^{lk}(\underline{a}) < r^{lk}(L)\}.$$

By the definition of $\Theta^{lk}_L(\underline{a})$, if $\underline{a} \in Y^{lk}$ then $\Theta^{lk}_L(\underline{a}) = 0$. Conversely, let $\Theta^{lk}_L(\underline{a}) = 0$. Suppose $r^{lk}(\underline{a}) = r^{lk}(L)$. Then by Lemma 2.1, $(S^{lk}(\underline{a}), T^{lk}(\underline{a})) = 0$, since $S^{lk}(\underline{a})\xi + T^{lk}(\underline{a})\zeta = 0$ if and only if $\zeta \in \operatorname{Ker}\operatorname{ad}^{r^{lk}(\underline{a})} T^{lk}(\underline{a}) S^{lk}(\underline{a})$. By assumption, $\Theta^{lk}_L(\underline{a}) = 0$, thus $\operatorname{Ker}\Theta^{lk}_L(\underline{a})$ is whole space. Therefore $\operatorname{rk} T^{lk}(\underline{a}) = 0$, which leads to a contradiction. Conclusion is fact that $\underline{a} \in Y^{lk}$ if and only if $\Theta^{lk}_L(\underline{a}) = 0$, and thus Y^{lk} is a closed subset of L nowhere dense in L. Let

$$Z^{lk} := Y^{lk} \cup \{\underline{a} \in L : \, d^{lk}(\underline{a}) < d^{lk}(L)\}.$$

If $\underline{a} \in L \setminus Z^{lk}$, then $r^{lk}(\underline{a}) = r^{lk}(L)$ and $d^{lk}(\underline{a}) = d^{lk}(L)$. Let

$$D^k := L \setminus \bigcup_{l>k} Z^{lk}.$$

Then D^k is a dense subset of L.

Lemma 2.4([5]). *For all $\underline{a}, \underline{a}' \in D^k$, $H_{\underline{a}}(k) = H_{\underline{a}'}(k)$ and $l(\underline{a}, k) = l(\underline{a}', k)$.*

For the proof of Lemma 2.4, see Lemma 5.3 in [5].

Definition 2.2. *The Generic Hilbert-Samuel function.* Put $H_L(k) := H_{\underline{a}}(k)$ and $l(L,k) := l(\underline{a},k)$, where $\underline{a} \in D^k$. We call $H_L(k)$ the generic Hilbert-Samuel function, and we call $l(L,k)$ the generic Chevalley estimate.

Assume that $\overline{L}$ lies in a coordinate chart for M^s. Take $\underline{a} \in L$. Let $k(\underline{a})$ be the least k, for which $H_{\underline{a}}(l)$ coincides with polynomial in l for $l > k$. Each vertex of diagram $\mathfrak{N}_{\underline{a}} = \mathfrak{N}(\mathcal{R}_{\underline{a}})$ has the order less than or equal to $k(\underline{a})$. If $k \geq k(\underline{a})$ and $l \geq l(\underline{a},k)$, then $\mathcal{R}^k(\underline{a}) = \operatorname{Ker}\Theta^{lk}(\underline{a})$ and by Corollary 2.1

$$\operatorname{Ker}\Theta^{lk} \cap J^k(\underline{\varphi}(\underline{a}))^{\mathfrak{N}_{\underline{a}}} = 0,$$

$\dim J^k(\underline{\varphi}(\underline{a}))^{\mathfrak{N}_{\underline{a}}} = \operatorname{rk}\Theta^{lk}(\underline{a})$. Thus there exists a nonzero minor $Q(\underline{a})$ of $\Theta^{lk}(\underline{a})$ restricted to $J^k(\underline{\varphi}(\underline{a}))^{\mathfrak{N}_{\underline{a}}}$ such that order of $Q(\underline{a})$ is $\operatorname{rk}\Theta^{lk}(\underline{a})$. Let $\xi = (\xi_\beta)_{|\beta| \leq k}$. Then by Cramer's rule

$$(1)\ \ \Theta^{lk}(\underline{a})\xi = 0$$

iff

$$\xi_\gamma - \sum_{\beta \notin \Delta(\underline{a}), |\beta| \leq k} \xi_\beta \frac{P^\beta_\gamma(\underline{a})}{Q(\underline{a})} = 0,$$

$\gamma \in \Delta(\underline{a})$, $|\gamma| \leq k$, where $\Delta(\underline{a}) := \mathbb{N} \setminus \mathfrak{N}_{\underline{a}}$ and $P^\beta_\gamma(\underline{a})$ are the minors from the system of equations (1). Let $\mathcal{B}(\mathfrak{N}_{\underline{a}}) := \{\alpha_1, \ldots, \alpha_t\}$ be the set of vertices of $\mathfrak{N}_{\underline{a}}$. Then

$$(2)\ \ G_i = (y - \underline{\varphi}(\underline{a})^{\alpha_i}) + \sum_{\gamma \in \Delta(\underline{a})} g_{i\gamma} \cdot (y - \underline{\varphi}(\underline{a}))^\gamma,\ i = 1, \ldots, t,$$

is the standard basis of $\mathcal{R}_{\underline{a}}$ and

$$g_{i\gamma} = \frac{P^{\alpha_i}_\gamma(\underline{a})}{Q(\underline{a})},$$

for all $\gamma \in \Delta(\underline{a})$, $|\gamma| \leq k$.

Let k be the largest integer such that $H_L(l)$ coincides with polynomial for $l \geq k$, and let $l = l(L,k)$. Thus all vertices of $\mathfrak{N}_{\underline{a}}$ has the order less than or equal to k for $\underline{a} \in D^k$. Therefore the set of diagrams on D^k is finite and thus has a minimum. Let $\underline{a} \in D^k$ be an element that $\mathfrak{N}_{\underline{a}}$ is the smallest diagram on D^k. Let

$$Z_Q := \{\underline{a}' : Q(\underline{a}') = 0\}.$$

Since Q is a quasianalytic function, Z_Q is a closed quasianalytic subset in L. We have the following

Lemma 2.5. *For all* $\underline{a}' \in D^k \setminus Z_Q$, $\mathfrak{N}_{\underline{a}'} = \mathfrak{N}_{\underline{a}}$.

Proof. By (2), there exist relations $G^{i'} \in \mathcal{R}_{\underline{a}'}$, such that

$$G_i' = (y - \underline{\varphi}(\underline{a}'))^{\alpha_i} + \sum_{\gamma \in \Delta(\underline{a}'), |\gamma| \leq k} g_{i\gamma}' \cdot (y - \underline{\varphi}(\underline{a}))^{\gamma} + M_i,$$

where $g_{i\gamma}' = \frac{P_{\alpha}^{\alpha_i}(\underline{a})}{Q(\underline{a})}$ and $M_i \in \widehat{m_{b'}}^{k+1}$. We took $\underline{a}$ such that $\mathfrak{N}_{\underline{a}} \leq \mathfrak{N}_{\underline{a}'}$, thus $\alpha_i \in \mathfrak{N}_{\underline{a}'}$ for each i. Therefore $\mathfrak{N}_{\underline{a}} \subset \mathfrak{N}_{\underline{a}'}$, which implicates $\mathfrak{N}_{\underline{a}'} \leq \mathfrak{N}_{\underline{a}}$. Finally $\mathfrak{N}_{\underline{a}} = \mathfrak{N}_{\underline{a}'}$, which ends the proof. □

Definition 2.3. We write $\mathfrak{N}_L := \mathfrak{N}_{\underline{a}}$, for all $\underline{a} \in D^k \setminus Z_Q$. We call $\mathfrak{N}_L$ the generic diagram of initial exponents for L.

Lemma 2.6. *([5], Lemma 5.8) For all $\underline{a} \in L$, $\mathfrak{N}_{\underline{a}} \geq \mathfrak{N}_L$.*

Chapter 3

Proof of implication $(2) \Rightarrow (3)$.

As we mentioned in the Introduction, to establish the semicontinuity of the diagram of initial exponents, E. Birstone and P. Milman proved simultaneously that the sets $Z_{\mathfrak{N}}^{+}$ (see Definition 3.2 below) are subanalytic and closed, if the uniform Chevalley estimate holds ([5], Proposition 8.6). Their proof cannot be directly applied in the quasianalytic settings, since it relies on the fact that the ring of formal power series is faithfully flat over the ring of analytic function germs, which is no longer available in the quasianalytic case. Here we are forced to follow a different, not so direct strategy. In this chapter, we prove that the sets $Z_{\mathfrak{N}}^{+}$ are quasi-subanalytic if the umniform Chevalley estimate holds (Proposition 3.3), and next the implication $(2) \Rightarrow (3)$. In the next chapter, we shall prove that the sets $Z_{\mathfrak{N}}^{+}$ are closed if the uniform Chevalley estimate holds (Theorem 4.1). In the proof we shall apply the results of this chapter, but also we shall develop a new approach, which consists in reducing the analysis of the diagram of initial exponents to quasi-subanalytic arcs. The foregoing two results together yield the semicontinuity of the diagram of initial exponents.

We should note that Bierstone–Milman's proof of Proposition 8.6 from paper [5] contains an error. In this chapter, we give an example (Remark 3.2) which shows why their proof is not completely correct and provide a proof of Proposition 3.3 which improves their arguments.

Assume that $Z \subset X$ are closed quasi-subanalytic sets. Let $\mathfrak{N} \in \mathcal{D}(n)$ and $\alpha \in \mathbb{N}^n$. We repeat the following notation by Bierstone and Milman:

Definition 3.1 Set

$$\mathfrak{N}(\alpha) := \mathbb{N}^n + \{\beta \in \mathfrak{N} : \beta \leq \alpha\}$$
$$\mathfrak{N}^{-}(\alpha) := \mathbb{N}^n + \{\beta \in \mathfrak{N} : \beta < \alpha\}.$$

Definition 3.2 Put

$$Z_{\mathfrak{N}}(\alpha) := Z \cup \{b \in X \setminus Z : \mathfrak{N}_b(\alpha) \geq \mathfrak{N}(\alpha)\},$$
$$Z_{\mathfrak{N}}^{+}(\alpha) := Z \cup \{b \in X \setminus Z : \mathfrak{N}_b(\alpha) > \mathfrak{N}(\alpha)\},$$
$$Z_{\mathfrak{N}} := Z \cup \{b \in X \setminus Z : \mathfrak{N}_b \geq \mathfrak{N}\},$$
$$Z_{\mathfrak{N}}^{+} := Z \cup \{b \in X \setminus Z : \mathfrak{N}_b > \mathfrak{N}\}.$$

We have the following

Lemma 3.1 *Let $\mathfrak{N}_1, \mathfrak{N}_2 \in \mathcal{D}(n)$ and $\alpha \in \mathbb{N}^n$. The following conditions are equivalent:*

(1) $\mathfrak{N}_1(\alpha) < \mathfrak{N}_2(\alpha)$,

(2) *there exists* $\theta \leq \alpha$ *such that* $\mathfrak{N}_1^-(\theta) = \mathfrak{N}_2^-(\theta)$, $\theta \in \mathfrak{N}_1(\theta)$ *and* $\theta \notin \mathfrak{N}_2(\theta)$.

Lemma 3.1. is an analogue of Remark 8.5 from [5], however authors do not give the proof, and thus we provide our own reasoning.

Proof. We prove (1)$\Rightarrow$(2). Let $\alpha_1, ..., \alpha_s$ be the vertices of $\mathfrak{N}_1(\alpha)$ and respectively $\beta_1, ..., \beta_t$ be the vertices of $\mathfrak{N}_2(\alpha)$. It is clear that for every $i = 1, ..., s$ $\alpha_i \leq \alpha$ and, for every $j = 1, ..., t$, $\beta_j \leq \alpha$. Let k be the largest number such that $\alpha_i = \beta_i$ if $i \leq k$. If $\alpha_i \neq \beta_i$ for each $i = 1, ..., \min\{t, s\}$ then we put $k = 0$. Since $\mathfrak{N}_1(\alpha) < \mathfrak{N}_2(\alpha)$, there exists a vertex α_{k+1} of diagram $\mathfrak{N}_1(\alpha)$ such that $\alpha_k < \alpha_{k+1} < \alpha$. We put $\theta = \alpha_{k+1}$. By the definition of k, there is no vertex β of $\mathfrak{N}_2(\alpha)$ such that $\beta_k < \beta_{k+1} < \alpha_{k+1}$. Therefore $\mathfrak{N}_1^{-1}(\theta) = \mathfrak{N}_2^-(\theta)$. Of course $\theta \in \mathfrak{N}_1(\theta)$ and $\theta \notin \mathfrak{N}_2(\theta)$.

Now we prove (2)$\Rightarrow$(1). Assume that there is $\theta \in \mathbb{N}^n$ such that $\mathfrak{N}_1^-(\theta) = \mathfrak{N}_2^-(\theta)$, $\theta \in \mathfrak{N}_1(\theta)$ and $\theta \notin \mathfrak{N}_2(\theta)$. Let $\alpha_1, ..., \alpha_k$ be the vertices of $\mathfrak{N}_1^-(\theta) = \mathfrak{N}_2^-(\theta)$. $\theta \notin \mathfrak{N}_2(\theta)$ thus θ cannot be generated by vertices $\alpha_1, ..., \alpha_k$. Therefore there exists a vertex $\alpha_{k+1} \leq \theta$ of the diagram $\mathfrak{N}_1(\theta)$ which generates θ. If $\mathfrak{N}_2(\alpha)$ does not have a vertex greater than α_{k+1} then $\mathfrak{N}_1(\alpha) < \mathfrak{N}_2(\alpha)$. If there exist a vertex β_{k+1} of $\mathfrak{N}_2(\alpha)$ it must be greater than α_{k+1}, because if $\beta_{k+1} < \alpha_{k+1}$ then $\mathfrak{N}_1^-(\theta) \neq \mathfrak{N}_2^-(\theta)$ and, if $\beta_{k+1} = \alpha_{k+1}$, then $\theta \in \mathfrak{N}_2^-(\theta)$, which is a contradiction. Therefore $\mathfrak{N}_1(\alpha) < \mathfrak{N}_2(\alpha)$. $\square$

Now we present two crucial propositions, which we need to show that uniform Chevalley estimate implies stratification by the diagram of initial exponents. We have the following

Proposition 3.1. *Assume that X has the uniform Chevalley estimate relatively to Z and let $Y \subset X$ be a closed quasi-subanalytic set such that $\mathfrak{N}_b^-(\alpha)$ is constant on Y for some $\alpha \in \mathbb{N}^n$. Then $Z \cup \{b \in Y \setminus Z : \alpha \notin \mathfrak{N}_b\}$ is a quasi-subanalytic set.*

To proof Proposition 3.1 we repeat the reasoning of the first part of the proof of Proposition 8.3 in [5], where the authors prove that the set considered is subanalytic.

Proof. If $\alpha \in \mathfrak{N}_b^-(\alpha)$ then $\{b \in Y : \alpha \notin \mathfrak{N}_b\} = \emptyset$, therefore it is enough to consider the case, where $\alpha \notin \mathfrak{N}_b^-(\alpha)$. We assume that X is compact.

Let $\varphi : M \to \mathbb{R}^n$ be a proper quasianalytic mapping such that $\varphi(M) = X$ (Corollary 1.1). By the assumption and Lemma 1.3, there exists a function $l_{\varphi^*} : \mathbb{N} \to \mathbb{N}$ such that $l_{\varphi^*}(b, k) \leq l_{\varphi^*}(k)$ for every $b \in X \setminus Z$ and $k \in \mathbb{N}$.

Let $k = |\alpha|$, $l = l_{\varphi^*}(k)$ and $b \in X$. Let $J^l(b)^{\mathfrak{N}^-(\alpha)} := \{\xi = (\xi_\beta)_{|\beta| \leq l} : \beta \in \mathfrak{N}^-(\alpha) \Rightarrow \xi_\beta = 0\}$, where $\mathfrak{N}^-(\alpha) = \mathfrak{N}_b^-(\alpha)$ for any $b \in Y \setminus Z$. Put

$$\mathfrak{N}^-(\alpha)_+ := \mathfrak{N}^-(\alpha) \cup \{\beta \in \mathbb{N}^n : \beta > \alpha\}.$$

We obtain a direct-sum decomposition

$$J^l(b)^{\mathfrak{N}^-(\alpha)} = J^l(b)^{\mathfrak{N}^-(\alpha)_+} \oplus (\widehat{m_y}^{>\alpha} \cap J^l(b)^{\mathfrak{N}^-(\alpha)}),$$

where $\widehat{m_b}^{>\alpha} \subset \widehat{m_b}$ is the ideal generated by monomials $(y-b)^\beta$ for $\beta > \alpha$. For $\xi \in J^l(b)^{\mathfrak{N}^-(\alpha)}$, we write $\xi = (\eta, \zeta)$ in order to the above decomposition.

Consider $a \in \varphi^{-1}(b)$ in local coordinate chart $(x_1, ..., x_m)$ in a neighborhood of a in M. Then we treat $J^l\varphi : J^l(b) \to J^l(a)$ as a matrix and we write $(A(a), B(a))$ for the matrix of $J^l\varphi|_{J^l(b)^{\mathfrak{N}^-(\alpha)}}$ according to the direct-sum above. We need two lemmas.

Lemma 3.2 *Let $b \in Y \setminus Z$. Then we have*

$$\alpha \notin \mathfrak{N}_b \Leftrightarrow \left[\left(\forall_{a\in\varphi^-(b)}\ A(a)\eta + B(a)\zeta = 0\right) \Rightarrow \eta = 0\right].$$

Proof. Consider $\xi \in J^l(b)^{\mathfrak{N}^-(\alpha)}$. Let $P_\xi = \sum_{|\beta|\le l} \xi_\beta (y-b)^\beta$ be a polynomial which generates ξ. If $\xi = (\eta, \zeta)$ we have $P_\xi = P_\eta + P_\zeta$ with respect to the direct-sum decomposition. Let $a \in \varphi^{-1}(b)$. Then $A(a)\eta + B(a)\zeta = 0$ if and only if $\varphi_a^*(P_\xi) \in m_a^{l+1}$.

Suppose $\alpha \notin \mathfrak{N}_b$ and let $\xi = (\eta, \zeta) \in J^l(b)^{\mathfrak{N}^-(\alpha)}$. Suppose that $A(a)\eta + B(a)\zeta = 0$ for all $a \in \varphi^{-1}(b)$. Then $\varphi_a^*(P_\xi) \in m_a^{l+1}$ and thus $\nu_{M,a}(\widehat{\varphi}_a^*(P)) > l$ for all $a \in \varphi^{-1}(b)$. By Chevalley estimate and Lemma 1.3, $\nu_{X.b}(P_\xi) > k$, therefore $P_\xi \in \mathcal{R}_b + (y-b)^{k+1}$. Since $|\alpha| \le k$, $(y-b)^{k+1} \subset (y-b)^{>\alpha}$. According to the definition of $P_\xi = P_\eta + P_\zeta$ and direct-sum decomposition we see that $P_\zeta \in (y-b)^{>\alpha}$ and thus $P_\eta \in \mathcal{R}_b + (y-b)^{>\alpha}$. By the assumption, $\eta \in J^l(b)^{\mathfrak{N}^-(\alpha)}$. Since $\alpha \notin \mathfrak{N}_b$, $\mathfrak{N}_b(\alpha) = \mathfrak{N}^-(\alpha)$ and $\mathfrak{N} \cup \{\beta : \beta > \alpha\} = \mathfrak{N}^-(\alpha)_+$. Thus $P_\eta \in \mathbb{R}[[y-b]]^{\mathfrak{N}_b \cup \{\beta:\beta>\alpha\}}$. On the other hand $\mathcal{R}_b + (y-b)^{>\alpha}$ is an ideal in $\mathbb{R}[[y-b]]$ whose diagram is $\mathfrak{N}_b \cup \{\beta : \beta > \alpha\}$. By Corollary 2.1 we get

$$\mathcal{R}_b + (y-b)^{>\alpha} \cap \mathbb{R}[[y-b]]^{\mathfrak{N}_b \cup \{\beta:\beta>\alpha\}} = 0.$$

Since P_η belongs to this intersection, we obtain $P_\eta = 0$. Therefore $\eta = 0$.

Suppose $\alpha \in \mathfrak{N}_b$. By Theorem 3.1, there exists $G \in \mathcal{R}_b$ such that $\operatorname{mon} G = (y-b)^\alpha$ and $G - (y-b)^\alpha \in \mathbb{R}[[y-b]]^{\mathfrak{N}_b}$. It is enough to show that there exists ξ such that, if $(A(a), B(a))\xi = 0$ for all $a \in \varphi^{-1}(b)$ then $\eta \neq 0$. Since $\mathfrak{N}_b^-(\alpha) = \mathfrak{N}^-(\alpha)$, $G - (y-b)^\alpha \in \mathbb{R}[[y-b]]^{\mathfrak{N}^-(\alpha)}$. Put $\xi = J^l G(b)$. Then $\xi \in J^l(b)^{\mathfrak{N}^-(\alpha)}$ and $(A(a), B(a))\xi = 0$ for all $a \in \varphi^{-1}(b)$ since $G \in \mathcal{R}_b$. By the definition of $\xi = (\eta, \zeta)$, we have $\xi_\alpha = 1$ and $\alpha \notin \mathfrak{N}^-(\alpha) \cup \{\beta : \beta > \alpha\}$, therefore $\eta \neq 0$. □

Lemma 3.3 *Let $\{C(\lambda) : \lambda \in \Lambda\}$ be a set of matrices from each of which has p columns and $\sharp\Lambda \ge p$. Let $Ker C(\lambda) = \{\xi = (\xi_1, \dots, \xi_p) : C(\lambda)\xi = 0 \text{ for all } \lambda \in \Lambda\}$. Then there exists $J \subset \Lambda$ such that $\sharp J = p$ and $Ker C(J) = Ker C(\lambda)$.*

Proof. If $\lambda \in \Lambda$, then $C(\lambda)\xi = 0$ if and only if for any row w of $C(\lambda)$ the scalar product $w \cdot \xi = 0$. Since the number of linearly independent rows from all $C(\lambda)$ is less than or equal to p, there exists such J. □

To complete the proof of proposition it is enough to show that $\Sigma = \{b \in Y \setminus Z : \alpha \notin \mathfrak{N}_b\}$ is a quasi-subanalytic set. Let $q = \binom{n+l}{l}$. For $\underline{a} = (a^1, \dots, a^q) \in M_\varphi^q$ we write

$$(\underline{A}(\underline{a}), \underline{B}(\underline{a})) := \begin{bmatrix} A(a^1) & B(a^1) \\ A(a^2) & B(a^2) \\ \vdots & \vdots \\ A(a^q) & B(a^q) \end{bmatrix}.$$

By Lemma 3.2, for all $b \in X$, there exits $\underline{a} \in \underline{\varphi}^{-1}(b) \subset M_\varphi^q$ such that $A(a)\eta + B(a)\zeta = 0$ for all $a \in \varphi^{-1}(b)$ if and only if $\underline{A}(\underline{a})\eta + \underline{B}(\underline{a})\zeta = 0$. Let $b \in Y \setminus Z$. By Lemma 3.2, $b \in \Sigma$ if and only if there exists $\underline{a} \in \underline{\varphi}^{-1}(b)$ such that

$$\underline{A}(\underline{a})\eta + \underline{B}(\underline{a})\zeta = 0 \Longleftrightarrow \eta = 0.$$

Let $\underline{a} \in M_\varphi^q$ and let $r(\underline{a}) := \operatorname{rank}(\underline{B}(\underline{a}))$. Then $r(\underline{a}) \le q$. Put $\underline{T} = \operatorname{ad}^{r(\underline{a})}\underline{B}(\underline{a}) \cdot \underline{A}(\underline{a})$, then

$$\operatorname{Ker} \underline{T}(\underline{a}) = \{\eta : \underline{A}(\underline{a})\eta \in \operatorname{Im} \underline{B}(\underline{a})\}.$$

We decompose $M_\varphi^q = \bigcup_r S^r$, where $S^r := \{\underline{a} \in M_\varphi^q : \operatorname{rank} \underline{B}(\underline{a}) = r\}$. We observe that each S^r is a difference of two analytic sets in M_φ^q— $S^r = W^r \setminus W^{r-1}$, where $W^r := \{\underline{a} \in M_\varphi^q : \operatorname{rank} \underline{B}(\underline{a}) \le r\}$. For each r, we put $S_0^r := \{\underline{a} \in S^r : \operatorname{Ker} T = 0\}$ and $S_0 := \bigcup_r S_0^r$, which is a quasi-subanalytic set. Since $\operatorname{Ker} T = 0$ if and only if $\underline{A}(\underline{a})(\eta) + \underline{B}(\underline{a})(\zeta) = 0$ implies $\eta = 0$ and, by Lemma 3.2, $\Sigma = Y \setminus Z \cup \underline{\varphi}(S_0)$, Σ is quasi-subanalytic. □

Proposition 3.2 *Assume that $Z \subset X$ are closed quasi-subanalytic sets such that X has the uniform Chevalley estimate relatively to Z. Let $\mathfrak{N} \in \mathcal{D}(n)$ and $\alpha \in \mathbb{N}^n$. Then $Z_{\mathfrak{N}}(\alpha)$ and $Z_{\mathfrak{N}}^+(\alpha)$ are quasi-subanalytic subsets of X.*

Remark 3.1 Proposition 3.2 is a weaker version of Proposition 8.6 from [5]. In comparison to the original proposition we cannot obtain a closedness of the set considered in the same way as in [5], because of lack of faithfull flatness of $\widehat{\mathcal{O}}_b$ over $\mathcal{O}_b$ in Q-analytic case. We provide a proof based on original reasoning, however we need to do some changes to make it correct. We show in Remark 3.2 after the proof that there is an error in original proof. We show an counterexample for this reasoning and point out the difference in our reasoning which makes the proof correct. In order to prove that $Z_{\mathfrak{N}}$ is closed, which is an assertion of Theorem 4.2, we develop a different method than in [5], in particular we reduce a problem to the case, where Y is an quasi-subanalytic arc.

Proof. We prove Proposition 332 by the induction on α. First assume that $\alpha = 0$. Then $\mathfrak{N}_b(\alpha) = \emptyset$ for $b \in X$, and there are two possible cases. If $\alpha \notin \mathfrak{N}$ then $\mathfrak{N}(\alpha) = \emptyset$, whence $Z_{\mathfrak{N}}(\alpha) = X$ and $Z_{\mathfrak{N}}^+(\alpha) = Z$. On the other hand, if $\alpha \in \mathfrak{N}$ then $\mathfrak{N}(\alpha) = \mathbb{N}^n$ and therefore $Z_{\mathfrak{N}}(\alpha) = Z_{\mathfrak{N}}^+(\alpha) = X$.

Now assume that the conclusion is true for all exponents $< \alpha$. Let $\beta_b \in \mathfrak{N}_b$ be the largest element which is less then α for $b \in X \setminus Z$, and $\beta_1 \in \mathfrak{N}$ be the largest element less than α in $\mathfrak{N}$. Let $\beta := \max\{\max\{\beta_b, b \in X \setminus Z\}, \beta_1\}$. Consider the following sets

$$\begin{aligned} X_1 &= Z_{\mathfrak{N}}(\alpha), \\ X_0 &= Z_{\mathfrak{N}}(\beta), \\ Z_1 &= Z_{\mathfrak{N}}^{+}(\alpha), \\ Z_0 &= Z_{\mathfrak{N}}^{+}(\beta). \end{aligned}$$

We shall prove that $Z_0 \subset Z_1 \subset X_1 \subset X_0$.

To show inclusion $Z_0 \subset Z_1$ take $b \in \{X \setminus Z : \mathfrak{N}_b(\beta) > \mathfrak{N}(\beta)\}$. By Lemma 3.1, there exists $\theta \leq \beta$ such that $\mathfrak{N}_b^-(\theta) = \mathfrak{N}^-(\theta)$, $\theta \in \mathfrak{N}(\theta)$ and $\theta \notin \mathfrak{N}_b$. Since $\theta \leq \beta < \alpha$ and again by Lemma 3.1, $\mathfrak{N}_b(\alpha) > \mathfrak{N}(\alpha)$. Therefore $\{b \in X \setminus Z : \mathfrak{N}_b(\beta) > \mathfrak{N}_b(\beta)\} \subset \{b \in X \setminus Z : \mathfrak{N}_b(\alpha) > \mathfrak{N}_b(\alpha)\}$ and $Z_0 \subset Z_1$.

It is clear that $Z_1 \subset X_1$. To prove that $X_1 \subset X_0$ it is enough to show that $\mathfrak{N}_b(\alpha) \geq \mathfrak{N}_b(\alpha)$ implies $\mathfrak{N}_b(\beta) \geq \mathfrak{N}_b(\beta)$. Suppose $\mathfrak{N}_b(\beta) < \mathfrak{N}(\beta)$. By Lemma 3.1, there exists $\theta \leq \beta$ such that $\mathfrak{N}_b^-(\theta) = \mathfrak{N}^-(\theta)$, $\theta \in \mathfrak{N}_b(\theta)$ and $\theta \notin \mathfrak{N}$. Since $\theta \leq \beta < \alpha$ and by Lemma 3.1, $\mathfrak{N}_b(\alpha) < \mathfrak{N}(\alpha)$. A contradiction, which proves that $X_1 \subset X_0$.

By the induction, X_0 and Z_0 are quasi-subanalytic sets. We prove our thesis for α.

Case 1. Assume $\alpha \in \mathfrak{N}^-(\alpha)$. Then $\mathfrak{N}(\alpha) = \mathfrak{N}^-(\alpha) = \mathfrak{N}(\beta)$. Let $b \in X$. If $\mathfrak{N}_b(\beta) = \mathfrak{N}(\beta)$, then $\alpha \in \mathfrak{N}_b(\beta)$, $\alpha \in \mathfrak{N}(\beta)$. Also $\mathfrak{N}_b(\beta) = \mathfrak{N}_b(\alpha)$ and $\mathfrak{N}(\beta) = \mathfrak{N}(\alpha)$. Thus $\mathfrak{N}_b(\alpha) = \mathfrak{N}(\alpha)$. On the other hand, if $\mathfrak{N}_b(\beta) > \mathfrak{N}(\beta)$ then $\mathfrak{N}_b(\alpha) > \mathfrak{N}(\alpha)$. Since $\alpha \in \mathfrak{N}_b \cap \mathfrak{N}$, $\mathfrak{N}_b(\alpha) \geq \mathfrak{N}(\alpha)$ if and only if $\mathfrak{N}_b(\beta) \geq \mathfrak{N}(\beta)$, and thus $X_1 = X_0$, which is quasi-subanalytic.

If $\mathfrak{N}_b(\beta) > \mathfrak{N}(\beta)$, then by Lemma 3.1 there exists $\theta \leq \alpha$ such that $\mathfrak{N}_b^-(\theta) = \mathfrak{N}^-(\theta)$, $\theta \in \mathfrak{N}(\theta)$ and $\theta \notin \mathfrak{N}_b(\theta)$. If $\theta \leq \beta < \alpha$, then $\mathfrak{N}_b(\alpha) > \mathfrak{N}(\alpha)$. If $\beta < \theta$, then $\theta = \alpha$ by the definition of β. Thus $\mathfrak{N}_b^-(\alpha) = \mathfrak{N}^-(\alpha)$, $\alpha \in \mathfrak{N}(\alpha)$ and $\alpha \notin \mathfrak{N}_b(\alpha)$. On the other hand $\alpha \in \mathfrak{N}^-(\alpha) = \mathfrak{N}_b^-(\alpha)$, which is a contradiction. Therefore, $\mathfrak{N}_b(\alpha)\, n(\alpha)$ if and only if $\mathfrak{N}_b(\beta) > \mathfrak{N}(\beta)$, and thus $Z_1 = Z_0$.

Case 2. Assume that $\alpha \notin \mathfrak{N}^-(\alpha)$ and $\alpha \in \mathfrak{N}$. Let $b \in X$. If $\mathfrak{N}_b(\beta) = \mathfrak{N}(\beta)$ and $\alpha \in \mathfrak{N}_b$, then $\mathfrak{N}_b(\alpha) \geq \mathfrak{N}(\alpha)$, since α is a vertex of $\mathfrak{N}$. If $\alpha \notin \mathfrak{N}_b$ then $\mathfrak{N}_b(\alpha) \geq \mathfrak{N}(\alpha)$, since $\mathfrak{N}(\alpha)$ has one additional vertex in comparison to $\mathfrak{N}_b(\alpha)$ and $\mathfrak{N}_b^-(\alpha) = \mathfrak{N}^-(\alpha)$. Therefore $\mathfrak{N}_b(\alpha) \geq \mathfrak{N}(\alpha)$ if and only if $\mathfrak{N}_b(\beta) \geq \mathfrak{N}(\beta)$ and $X_1 = X_0$.

If $\mathfrak{N}_b(\alpha) > \mathfrak{N}(\alpha)$, then there exists $\theta < \alpha$ such that $\mathfrak{N}_b^-(\theta) = \mathfrak{N}^-(\theta)$, $\theta \in \mathfrak{N}(\theta)$ and $\theta \notin \mathfrak{N}_b$. If one can find $\theta < \beta$, then $\mathfrak{N}_b(\beta) > \mathfrak{N}(\beta)$. If not, then $\theta = \alpha$ and $\mathfrak{N}_b(\beta) = \mathfrak{N}(\beta)$, $\theta \notin \mathfrak{N}_b$ and $\theta \in \mathfrak{N}$. If we assume that $\mathfrak{N}_b(\beta) = \mathfrak{N}(\beta)$, $\alpha \notin \mathfrak{N}_b$ and $\alpha \in \mathfrak{N}$, then by Lemma 3.1 $\mathfrak{N}_b(\alpha) > \mathfrak{N}(\alpha)$. Therefore $\mathfrak{N}_b(\alpha) > \mathfrak{N}(\alpha)$ if and only if either $\mathfrak{N}_b(\beta) > \mathfrak{N}(\beta)$ or $\mathfrak{N}_b(\beta) = \mathfrak{N}(\beta)$, $\alpha \in \mathfrak{N}$ and $\alpha \notin \mathfrak{N}_b$. In that case $Z_1 = Z_0 \cup \{b \in X_0 \setminus Z_0 : \alpha \notin \mathfrak{N}_b\}$. Z_0 is quasi-subanalytic by induction, $\{b \in X_0 \setminus Z_0 : \alpha \notin \mathfrak{N}_b\}$ is quasi-subanalytic by Proposition 3.1, and thus Z_1 is quasi-subanalytic.

Case 3. Assume $\alpha \notin \mathfrak{N}^-(\alpha)$ and $\alpha \notin \mathfrak{N}$. If $\mathfrak{N}_b(\alpha) > \mathfrak{N}(\alpha)$ then there exists $\theta \leq \alpha$ such that $\mathfrak{N}^-(\theta) = \mathfrak{N}_b^-(\theta)$, $\theta \notin \mathfrak{N}_b$ and $\theta \in \mathfrak{N}$. If $\theta \leq \beta$ then

$\mathfrak{N}_b(\beta) > \mathfrak{N}(\beta)$. If not, thus $\theta = \alpha \in \mathfrak{N}$, which is a contradiction. Therefore $\mathfrak{N}_b(\alpha) > \mathfrak{N}(\alpha)$ if and only if $\mathfrak{N}_b(\beta) > \mathfrak{N}(\beta)$ and $Z_1 = Z_0$.

If $\mathfrak{N}_b(\alpha) \geq \mathfrak{N}(\alpha)$, then either $\mathfrak{N}_b(\alpha) > \mathfrak{N}(\alpha)$ or $\mathfrak{N}_b(\alpha) = \mathfrak{N}(\alpha)$. Thus $\alpha \notin \mathfrak{N}_b$ and $\mathfrak{N}_b(\beta) = \mathfrak{N}(\beta)$ or $\alpha \in \mathfrak{N}_b$ and $\mathfrak{N}_b(\alpha) > \mathfrak{N}(\alpha)$. Therefore $\mathfrak{N}_b(\alpha) \geq \mathfrak{N}(\alpha)$ if and only if $\mathfrak{N}_b(\beta) > \mathfrak{N}(\beta)$ or $\mathfrak{N}_b(\beta) = \mathfrak{N}(\beta)$ and $\alpha \notin \mathfrak{N}_b$. We get $X_1 = Z_1 \cup \{b \in X_0 \setminus Z_1 : \alpha \notin \mathfrak{N}_b\}$, which is quasi-subanalytic via Case 1, Case 2 and Proposition 3.1. This ends the proof. □

Remark 3.2 In the original proof authors set β–the largest element of $\mathfrak{N}$, which is smaller than α. Then, in Case 2, they claim that if $\alpha \notin \mathfrak{N}^-(\alpha)$, then $\mathfrak{N}_b(\alpha) > \mathfrak{N}(\alpha)$ if and only if either $\mathfrak{N}_b(\beta) > \mathfrak{N}(\beta)$ or $\mathfrak{N}_b(\beta) = \mathfrak{N}(\beta)$ and $\alpha \in \mathfrak{N}$, $\alpha \notin \mathfrak{N}_b$. We give an example that this equivalence is not true.

Consider $\mathfrak{N}_b := \{(0,0,1),(0,1,0)\} + \mathbb{N}^3$, $\mathfrak{N} := \{(1,0,0),(0,0,1)\} + \mathbb{N}^3$. Let $\alpha = (1,0,0)$. It is clear that $\mathfrak{N}_b(\alpha) < \mathfrak{N}(\alpha)$. The largest element in $\mathfrak{N}$ less then α is $\beta = (0,0,1)$, thus $\mathfrak{N}_b(\beta) = \mathfrak{N}(\beta)$. Of course $\alpha \in \mathfrak{N}$, $\alpha \notin \mathfrak{N}^-(\alpha)$ and $\alpha \notin \mathfrak{N}_b$. Therefore the equivalence above is not valid. Now if we take β as in our proof, then $\beta \geq (0,1,0)$, and this phenomenon does not occur.

Corollary 3.1 *Let $\mathfrak{N} \in \mathcal{D}(n)$. Then $Z^+_{\mathfrak{N}}$ is a quasi-subanalytic set.*

Proof. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_k = \alpha$ be the vertices of $\mathfrak{N}$ and $\beta_1 < \beta_2 < \cdots < \beta_l$ be the vertices of $\mathfrak{N}_b$ for $b \in X \setminus Z$. We will prove that $\mathfrak{N}_b > \mathfrak{N}$ if and only if $\mathfrak{N}_b(\alpha) > \mathfrak{N}(\alpha) = \mathfrak{N}$.

Assume that $\mathfrak{N}_b > \mathfrak{N}$. If there exists $s \leq l$ such that $\beta_s > \alpha_s$ and $\beta_i = \alpha_i$ for $i < s$, then, since $s \leq l$, all β_i and α_i for $i \leq s$ are vertices of $\mathfrak{N}_b(\alpha)$ and $\mathfrak{N}(\alpha)$ respectively. Therefore $\mathfrak{N}_b(\alpha) > \mathfrak{N}(\alpha)$. If there is no such s thus for all $i \leq l$ $\beta_i = \alpha_i$ for $i \leq l$ and $l < k$. In that case $\mathfrak{N}_b(\alpha) = \mathfrak{N}_b$, and since $\mathfrak{N}(\alpha) = \mathfrak{N}$ we get $\mathfrak{N}_b(\alpha) > \mathfrak{N}(\alpha)$.

Now assume that $\mathfrak{N}_b(\alpha) > \mathfrak{N}(\alpha)$. If there exists $s \leq l$ such that $\beta_s > \alpha_s$ and $\beta_i = \alpha_i$ for $i < s$ we get immediately $\mathfrak{N}_b > \mathfrak{N}$. If there is no such s, then $\mathfrak{N}_b(\alpha) = \mathfrak{N}_b$ and $\mathfrak{N}_b > \mathfrak{N}$, or there exists vertex $\beta_s > \alpha = \alpha_k$, $s \leq k$ and $\mathfrak{N}_b > \mathfrak{N}$. Therefore $Z^+_{\mathfrak{N}} = Z^+_{\mathfrak{N}}(\alpha)$, which is quasi-subanalytic by Proposition 3.2.

□

Theorem 3.1 *Let $Z \subset X$ be closed quasi-subanalytic sets such that X has the uniform Chevalley estimate relatively to Z. Then, for any compact set $K \subset X$, $\sharp\{\mathfrak{N}_b : b \in (X \setminus Z) \cap K\} < \infty$.*

Proof. We can assume that X is compact quasi-subanalytic set. Let Y be a closed quasi-subanalytic set such that $Z \subset Y \subset X$ and $\sharp\{\mathfrak{N}_b : b \in (X \setminus Y)\} < \infty$. Such a set always exists because we can take as Y whole $X \setminus Z$. We will prove that there exists closed quasi-subanalytic set $Y' \subset Y$ such that $\dim(Y' \setminus Z) < \dim(Y \setminus Z)$ and $\sharp\{\mathfrak{N}_b : b \in X \setminus Y'\} < \infty$.

Let $\varphi : M \to \mathbb{R}^n$ be a proper real analytic mapping such that $\varphi(M) = X$ and let $0 \neq s \in \mathbb{N}$. We denote by $\underline{\varphi}^s : M^s_\varphi \to \mathbb{R}^n$ the induced mapping from the s-fold fibre-product and we write $\underline{x} = (x_1, \ldots, x_s) \in M^s_\varphi$. For $\underline{x}$ we write

$\varphi^{-1}(\underline{\varphi}(\underline{x})) = \bigcup_{i=1}^{r(\underline{x})} S_i(\underline{x})$, where $S_i(\underline{x})$ are the distinct connected components of $\varphi^{-1}(\underline{\varphi}(\underline{x}))$. Let

$$\underline{M}^s_\varphi = \{\underline{x} \in M^s_\varphi | x_i \neq x_j \Leftrightarrow i \neq j \wedge \#\{x_1, \ldots, x_n\} \cap S_t(\underline{x}) \leq 1,\, 0 \leq t \leq r(\underline{x})\},$$

which means that each x_i lies in a distinct connected component of $\varphi^{-1}(\underline{\varphi}(\underline{x}))$. Note that if $\underline{x} \in \underline{M}^s_\varphi$, then $r(\underline{x}) \geq s$. By Corollary 1.7 $\underline{M}^s_\varphi$ is quasi-subanalytic. Let $L = \varphi^{-}(Y)$ and $\underline{L}^s = \underline{\varphi}^{-1}(Y) \cap \underline{M}^s_\varphi$. For each s we have a diagram of inclusions and projections:

$$\begin{array}{ccccc} \underline{L}^{s+1} & \subset & \underline{M}^{s+1}_\varphi & \subset & M^{s+1}_\varphi \\ \downarrow & & \downarrow & & \downarrow \\ \underline{L}^s & \subset & \underline{M}^s_\varphi & \subset & M^s_\varphi \\ \downarrow & & \downarrow & & \downarrow \\ L & \subset & M & \subset & M \end{array} ,$$

where down arrows represents the projections $\pi(x_1, \ldots, x_{s+1}) = (x_1, \ldots, x_s)$.

Consider a set $\underline{L}^s \setminus ((\underline{\varphi}^s)^{-1}(Z) \cup \pi(\underline{L}^{s+1}))$. This is a set of those $\underline{x} \in \underline{L}^s$ that $\underline{\varphi}^s(\underline{x}) \notin Z$ and $(\underline{\varphi}^s)^{-1}(\underline{x})$ has exactly s connected components. By Corollary 1.6, $\underline{L}^s \setminus ((\underline{\varphi}^s)^{-1}(Z) \cup \pi(\underline{L}^{s+1})) = \bigcup_j W_{s,j}$, where this sum is a finite partition into a smooth, connected and quasi-subanalytic sets. Since we assumed that X is compact and by Corollary 1.5, the number of connected components of the fibre $\varphi^{-1}(b)$, for $b \in X$, is bounded. Let t be the largest number of connected components of the fibre. Therefore

$$Y = Z \cup \bigcup_{s=1}^{t} \bigcup_j \underline{\varphi}^s(W_{s,j}).$$

For $\underline{a} \in W_{s,j}$ we have $\mathcal{R}_{\underline{a}} = \bigcap_{i=1}^s \operatorname{Ker} \widehat{\varphi}^*_{a_i} = \mathcal{F}_b(X)$, where $\underline{\varphi}^s(\underline{a}) = b$. Therefore $\mathfrak{N}_{\underline{a}} := \mathfrak{N}(\mathcal{R}_{\underline{a}}) = \mathfrak{N}_b$. For each s, j, there exists a generic diagram $\mathfrak{N}_{s,j} \in \mathcal{D}$ (see Chapter 2) such that $\mathfrak{N}_{\underline{a}} = \mathfrak{N}_{s,j}$ on an open and dense set in $W_{s,j}$, and $\mathfrak{N}_{\underline{a}} \geq \mathfrak{N}_{s,j}$. By Corollary 3.1, $Z^+_{\mathfrak{N}_{s,j}}$ is quasi-subanalytic set, thus $\underline{\varphi}^s(W_{s,j}) \setminus Z^+_{\mathfrak{N}_{s,j}}$ is quasi-subanalytic. For all $b \in \underline{\varphi}^s(W_{s,j}) \setminus Z^+_{\mathfrak{N}_{s,j}}$ we have $\mathfrak{N}_b = \mathfrak{N}_{s,j}$ and for all $b \in \underline{\varphi}^s(W_{s,j}) \cap Z^+_{\mathfrak{N}_{s,j}}$ we have $\mathfrak{N}_b > \mathfrak{N}_{s,j}$. We will prove that $\underline{\varphi}^s(W_{s,j}) \setminus Z^+_{\mathfrak{N}_{s,j}}$ is dense in $\overline{\underline{\varphi}^s(W_{s,j})}$.

Let $A_{s,j}$ be a subset of $W_{s,j}$ such that for all $\underline{a} \in A_{s,j}$, $\mathfrak{N}_{\underline{a}} > \mathfrak{N}_{s,j}$. Then $\underline{\varphi}^s(W_{s,j} \setminus A_{s,j}) = \underline{\varphi}^s(W_{s,j}) \setminus Z^+_{\mathfrak{N}_{s,j}}$. Since $\underline{\varphi}^s$ is continuous we have a sequence of inclusions

$$\underline{\varphi}^s(W_{s,j} \setminus A_{s,j}) \subset \underline{\varphi}^s(\overline{W_{s,j} \setminus A_{s,j}}) \subset \overline{\underline{\varphi}^s(W_{s,j} \setminus A_{s,j})},$$

and therefore

$$\overline{\underline{\varphi}^s(W_{s,j} \setminus A_{s,j})} \subset \overline{\underline{\varphi}^s(\overline{W_{s,j} \setminus A_{s,j}})} \subset \overline{\underline{\varphi}^s(W_{s,j} \setminus A_{s,j})}.$$

Since $\underline{\varphi}^s$ is proper, it is also closed, and thus

$$\overline{\underline{\varphi}^s(\overline{W_{s,j} \setminus A_{s,j}})} = \underline{\varphi}^s(\overline{W_{s,j} \setminus A_{s,j}}) = \overline{\underline{\varphi}^s(W_{s,j} \setminus A_{s,j})}.$$

Therefore $\underline{\varphi}^s(\overline{W_{s,j} \setminus A_{s,j}}) = \underline{\varphi}^s(\overline{W_{s,j}}) = \overline{\underline{\varphi}^s(W_{s,j}) \setminus Z^+_{\mathfrak{N}_{s,j}}}$. Now let

$$Z_{s,j} := \overline{\underline{\varphi}^s(W_{s,j})} \setminus \underline{\varphi}^s(W_{s,j}) \setminus Z^+_{\mathfrak{N}_{s,j}}.$$

Consider $Y' = Z \cup \bigcup_{s,j} \overline{Z_{s,j}}$. Since $\dim Z_{s,j} < \dim Y$ for each s, j we have $\dim Y' < \dim Y$, $Z \subset Y' \subset Y$ and Y' is quasi-subanalytic. □

Corollary 3.2 *If X has the uniform Chevalley estimate relatively to Z, then $Z_{\mathfrak{N}}$ is quasi-subanalytic set.*

Proof. We can adapt here the proof of Corollary 8.9 from [5]. We can assume that X is compact. Let α be greater or equal to the largest vertex of $\mathfrak{N}$ and $\mathfrak{N}_b$ of all $\mathfrak{N}_b$, $b \in X \setminus Z$. By Theorem 3.1, there exists such α. Therefore $\mathfrak{N}_b \geq \mathfrak{N}$ if and only if $\mathfrak{N}_b(\alpha) \geq \mathfrak{N}(\alpha)$, and thus $Z_{\mathfrak{N}} = Z_{\mathfrak{N}}(\alpha)$. □

Corollary 3.3 *Let $Z \subset X$ be closed quasi-subanalytic sets, such that X has the uniform Chevalley estimate relatively to Z. Then there is a stratification of X such that the diagram of initial exponents is constat on each stratum and Z is a sum of strata.*

Proof. We can assume that X is compact. Then there is a finite number of diagrams $\mathfrak{N}_1, \ldots, \mathfrak{N}_k$ of initial exponents on $X \setminus Z$. Since $Z_{\mathfrak{N}_i}$ and $Z^+_{\mathfrak{N}_i}$ are quasi-subanalytic for $i = 1, \ldots, k$, thus $Y_{\mathfrak{N}_i} := Z_{\mathfrak{N}_i} \setminus Z^+_{\mathfrak{N}_i}$ is also quasi-subanalytic. On the other hand $Y_{\mathfrak{N}_i} = \{b \in X \setminus Z : \mathfrak{N}_b = \mathfrak{N}_i\}$. Thus we can write

$$X = Z \cup \bigcup_{i=1}^{k} Y_{\mathfrak{N}_i}.$$

Whence there is a stratification of X compatible with Z and $\{Y_{\mathfrak{N}_i}\}_{i=1,\ldots,k}$, and the diagram of initial exponents is constant on each stratum outside Z. □

We proved that the uniform Chevalley estimate implies a stratification by the diagram of initial exponents. We shall prove that reverse implication also holds. We have

Theorem 3.2 *Suppose that X admits a stratification by the diagram of initial exponents such that Z is the union of strata. Then there is a uniform Chevalley estimate on X relatively to Z.*

The proof of Theorem 3.2 is the consequence of the two propositions below, which are quasianalytic analogues of Proposition 8.14 and proposition 8.15 from [5]. Let X be a closed quasi-subanalytic subset of $\mathbb{R}^n$.

Proposition 3.3 ([5], Proposition 8.14) *Suppose that $Y \subset X$ is a quasi-subanalytic set such that $\mathfrak{N}_b$ is constant on Y. Let $K \subset X$ be compact. Then there exist $l_K(k)$ such that $l_{\varphi_*}(b, k) \leq l_K(k)$ for all $b \in K \cap Y$.*

The proof of Proposition 3P3 can be easily reduced to the following

Proposition 3.4 ([5], Proposition 8.15) *Let $s \geq 1$ and $\underline{\varphi} : M^s_\varphi \to \mathbb{R}^n$. Let L be a relatively compact quasi-subanalytic subset of M^s_φ such that $\mathfrak{N}_{\underline{a}} = \mathfrak{N}(\mathcal{R}_{\underline{a}})$ is constant on L. Then there exists $l_L(k)$ such that $l_{\varphi^*}(\underline{a}, k) \leq l_L(k)$, for $\underline{a} \in L$.*

The proof of Proposition 3.4 is based on the analysis of jets and the systems of linear equations which coefficients are Q-analytic functions. The reason why it can be carried over from the analytic case is the fact that the analysis mentioned above are reduced to the properties of the ring of formal power series and several good properties which Q-analytic functions share with analytic functions, for instance the property of identity. Therefore we could just repeat the proof by E. Bierstone and P. Milman.

Chapter 4

Proof of implication $(2) \Rightarrow (4)$.

In this chapter we shall prove that the uniform Chevalley estimate implies the Zariski semi-continuity of the diagram of initial exponents. Let $Z \subset X$ be closed quasi-subanalytic sets in $\mathbb{R}^n$ such that X has the uniform Chevalley estimate relatively to Z. Let $\varphi : M \to \mathbb{R}^n$ be a proper Q-analytic mapping from a Q-analytic manifold M such that $\varphi(M) = X$. By Theorem 3.1, for any compact K, the set of diagrams of initial exponents is finite on $K \cap X$. Therefore, to prove Zariski semi-continuity it is sufficient to prove that the set

$$Z_{\mathfrak{N}} := Z \cup \{b \in X \setminus Z : \mathfrak{N}_b \geq \mathfrak{N}\}$$

is a closed quasi-subanalytic set. By Corollary 3.2, $Z_{\mathfrak{N}}$ is quasi-subanalytic, thus it remains to prove that $Z_{\mathfrak{N}}$ is closed.

In the paper [5] the authors proved that in the classical analytic case $Z_{\mathfrak{N}}$ is a closed subanalytic set. They used faithful flatness of the ring of formal power series over the ring of germs of analytic functions. In our reasoning we are forced to provide a different method, since it is not known if the ring $\mathbb{R}[[y-b]]$ of formal power series is faithfully flat over the ring $\mathcal{O}_b$ of germs of Q-analytic functions at b. This is an open problem related to the problem of noetherianity of $\mathcal{O}_b$, which has been widely studied for the past several decades, but remains unsolved.

Our proof relies on a reduction to the analysis only of quasi-subanalytic arcs, and on Proposition 4.1, which is a special case of Proposition 8.3 from [5] for closed quasi-subanalytic arcs.

In order to prove Proposition 4.1, we introduce a concept of an essential point $\underline{a} \in M^q_\varphi$ which determines the diagram $\mathfrak{N}_b$ with $\underline{\varphi}(\underline{a}) = b$ (Definition 4.2). We show that the set of essential points is definable. Then we apply curve selection to find a quasi-subanalytic arc $\tilde{L}$ lying over L and contained in the set of essential points. Next, by Puiseux's theorem, we are able to reduce the proof to the analysis of jets parameterized by Q-analytic functions of one variable. Since the local rings of quasianalytic functions of one variable are noetherian, we can find a quasianalytic solution to a system of linear equations, which describes when a multi-index α belongs to the diagram of initial exponents (Proposition 4.1).

Definition 4.1. Let M be a Q-analytic manifold. We say that $l : [0, \epsilon] \to M$ is a quasi-subanalytic arc if l is a continuous, definable and injective function.

Remark 4.1. It follows from the cell decomposition that a closed definable subset of pure dimension 1 is a finite sum of images of definable arcs. By abuse of terminology, by a definable arc we often mean both the arc and its image $L = l([0,\epsilon])$. In order to prove Proposition 4.1, we need a quasianalytic version of Puiseux's theorem stated below. It is a special case of the quasianalytic version of Puiseux's theorem with parameter due to K.J. Nowak([27]).

Let us notice that subanalytic arcs are analytic curves and their local analytic rings are noetherian. Yet the former is no longer true in quasianalytic structures, as shown by K.J. Nowak in the example constructed in paper [29]. The latter seems to be doubtful as well, being related to the failure of the following splitting problem posed by K.J. Nowak in papers [18] and [21]:

Let f be a Q-analytic function at $0 \in \mathbb{R}^k$ with Taylor series $\widehat{f}$. Split the set $\mathbb{N}^k$ of exponents into two disjoint subsets A and B, $\mathbb{N}^k = A \cup B$, and decompose the formal series $\widehat{f}$ into the sum of two formal series G and H, supported by A and B, respectively. Do there exist two Q-functions g and h at $0 \in \mathbb{R}^k$ with Taylor series G and H, respectively?

In some special cases of splitting the Taylor exponents, a negative answer was given by H. Sfouli [32].

Although quasi-subanalytic arc not need to be Q-analytic curve, it can be parameterized by a Q-analytic function. It is a consequence of the following

Theorem 4.1*(Puiseux's Theorem). Let*

$$f : (0,1) \to \mathbb{R}$$

be a bounded definable function. Then there exists an interval $I := (-\epsilon, \epsilon)$ such that

(1) either the function f vanishes on $I \cap (0,1)$;

(2) or there exist $r \in \mathbb{N}$ and a definable function $F(t)$, Q-analytic on I such that

$$f(t) = F(t^{1/r}), \quad for\, all\, t \in I \cap (0,1).$$

Our goal is the following

Theorem 4.2 *The set $Z_{\mathfrak{N}}$ is a closed quasi-subanalytic set.*

Proof. Let $\{U_c\}_{c>0}$ be the family of open balls with center at 0 in $\mathbb{R}^n$ and radius c. We can represent the set X in the following way

$$X = \bigcup_{c>0} X \cap \overline{U_c}.$$

It is clear that for every $c > 0$, $X \cap \overline{U_c}$ is a compact quasi-subanalytic set. The ideal $\mathcal{F}_b(X \cap \overline{U_c})$ for $b \in X \cap U_c$ coincides with the ideal $\mathcal{F}_b(X)$. Suppose that the conclusion of Theorem 4.2 is true for the compact definable sets. Then, for each $c > 0$, $Z_{\mathfrak{N}} \cap U_c$ is a trace of closed quasi-subanalytic set

$$Z_n^c = (Z \cap \overline{U_c}) \cup \{b \in (X \setminus Z) \cap \overline{U_c} : \mathfrak{N}_b \leq \mathfrak{N}\}.$$

Suppose that $\{b_k\}_{k\in\mathbb{N}} \subset Z_{\mathfrak{N}}$ and $\lim_{k\to\infty} b_k = b$. There exists $c > 0$ such that $b \in U_c$. Clearly $Z_{\mathfrak{N}} \cap U_c = Z^c_{\mathfrak{N}} \cap U_c$. There exists $K \in \mathbb{N}$ such that $b_k \in Z_{\mathfrak{N}} \cap U_c$ for $k > K$. Since $Z^c_{\mathfrak{N}}$ is closed, $b \in Z^c_{\mathfrak{N}} \cap U_c$ and thus $b \in Z_{\mathfrak{N}}$. Therefore we can assume that X is a compact quasi-subanalytic set.

Assume that X is compact. By Theorem 3.1 the set $\{\mathfrak{N}_b, b \in X \setminus Z\}$ is finite. By Corollary 4.3, $Z_{\mathfrak{N}}$ is a quasi-subanalytic set. Thus it remains to prove that $Z_{\mathfrak{N}}$ is closed.

Let $\mathcal{W}$ be a stratification of X such that $Z_{\mathfrak{N}}$ is a sum of strata and the diagram of initial exponents is constant on each stratum. Let $W \in \mathcal{W}$. It is enough to show, that

$$(*)\quad \mathfrak{N}_b \geq \mathfrak{N}_W \quad \text{for each } b \in W'$$

where $\mathfrak{N}_W$ is the diagram of initial exponents of the ideal of formal relations for each $y \in W$, and $W' \subset \overline{W} \setminus W$ is a stratum such that $W' \cap Z = \emptyset$. Take $b \in W'$. By the curve selection lemma ([11],Chap. 6, Corollary 1.5) there exists a continuous definable, injective function $f : (0, \epsilon) \to W$ such that $\lim_{x\to 0} f(x) = b$. It is then enough to prove $(*)$ for the quasi-subanalytic arcs with the end at W'. Theorem 4.2 will be proved once we establish the following

Proposition 4.1. *Let $L \subset X$ be a closed quasi-subanalytic arc. Let $\alpha \in \mathbb{N}^n$, and for each $b \in L$, $\mathfrak{N}_b(\alpha)^- = \mathfrak{N}(\alpha)$. Then the set $Z \cup \{b \in L \setminus Z : \alpha \notin \mathfrak{N}_b\}$ is closed quasi-subanalytic set.*

Indeed, let us assume, that for some quasi-subanalytic arc L with end $b \in W'$ we have $\mathfrak{N}_b < \mathfrak{N}_W$. Let $\beta_1, \ldots, \beta_s$ be the vertices of $\mathfrak{N}_b$, and let $\gamma_1, \ldots, \gamma_r$ be the vertices of $\mathfrak{N}_W$. Clearly $\mathfrak{N}_b < \mathfrak{N}_W$ if and only if $r < s$ and, for $i \leq r$, $\beta_i = \gamma_i$, or there exists $j \leq \min\{s, r\}$ such that $\beta_j < \gamma_j$ and $\beta_i = \gamma_i$ for $i < j$. Therefore, there exists $\alpha \in \mathfrak{N}_b$ such that $\mathfrak{N}_b(\alpha)^- = \mathfrak{N}_W(\alpha)^-$. Then $Z \cup \{b' \in L : \alpha \notin \mathfrak{N}_{b'}\} = Z \cup L \setminus \{b\}$, which is not closed. This contradicts the conclusion of Proposition 4.1.

Proof. (of Proposition 4.1.) By Proposition 3.2, $Z \cup \{b \in L : \alpha \notin \mathfrak{N}_b\}$ is quasi-subanalytic. It remains to prove, that it is a closed set. Let Σ be the set from the proof of Proposition 3.1 for $Y = L$. To prove that $Z \cup \Sigma$ is closed it is enough to show, that $(L \setminus Z) \setminus \Sigma$ is open in $L \setminus Z$. If $\alpha \in \mathfrak{N}$, then $(L \setminus Z) \setminus \Sigma$ would be an empty set and thus open. Whence we can assume that $\alpha \notin \mathfrak{N}$ and, since we only will consider multi-indices smaller then or equal to α, we can assume that $\mathfrak{N} = \mathfrak{N}^-(\alpha)$. Therefore $\mathfrak{N} \subset \mathfrak{N}_b$ for all $b \in L \setminus Z$.

Let $k = |\alpha|$ and $l = l_{\varphi^*}(k)$ (where $l_{\varphi^*}(k)$ is the uniform Chevalley estimate). Let q be the largest number of connected components of $\varphi^{-1}(b)$ for $b \in X$. Fix a point $b \in (L \setminus Z) \setminus \Sigma$. Obviously we can assume that b is the end of L. Let $\tau : [0, 1] \to L$ be the parametrization of L such that $\tau(0) = b$.

Consider a sequence $\{b_\omega\}_{\omega\in\mathbb{N}} \subset L$ with $\lim_{\omega\to\infty} b_\omega = b$. Such a sequence is a relatively compact set, because, by the assumption on X, L is compact. Since $\underline{\varphi}$ is a proper map, the set $\underline{\varphi}^{-1}(\{b_\omega\}_{\omega\in\mathbb{N}} \cup \{b\})$ is a compact subset of M^q_φ. By Lemma 4.1, $\underline{\varphi}(S) = X \setminus Z$, and thus we can take a sequence $\{\underline{a}_n\}_{n\in\mathbb{N}} \subset S$ such that $\underline{\varphi}(\underline{a}_\omega) = b_\omega$. Of course $\{\underline{a}_\omega\}_{\omega\in\mathbb{N}}$ has an accumulation point at $\underline{a} \in \overline{S}$ such that $\underline{\varphi}(\underline{a}) = b$. Therefore, there exists a quasi-subanalytic arc $\tilde{L} = \tilde{\tau}([0,1))$, where $\tilde{\tau} : [0,1) \to M_{\varphi}^{q}$ is a parametrization of $\tilde{L}$, such that $\tilde{\tau}(0) = \underline{a}$ and $\tilde{\tau}((0,1)) \subset S$. Consider the following diagram

$$\begin{tikzcd}
\underline{\varphi}\circ\tilde{\tau}:[0,1) \arrow[r] \arrow[dr, "\tau^{-1}\circ\underline{\varphi}\circ\tilde{\tau}"'] & L \\
& {[0,1)} \arrow[u, "\tau"']
\end{tikzcd}$$

By the monotonicity theorem ([11],Chap. 3, Theorem 1.2) and the fact that $\tau^{-1}\circ\underline{\varphi}\circ\tilde{\tau}$ is not constant, we can assume that $\tau^{-1}\circ\underline{\varphi}\circ\tilde{\tau}$ is a strictly increasing function on some interval $[0,p) \subset [0,1)$. Therefore $\underline{\varphi}(L') \supset \tau([0,p))$. By Puiseux's theorem, there exists a Q-analytic parametrization

$$\varepsilon : (-1,1) \to M_{\varphi}^{q}$$

such that $\varepsilon([0,1)) = \tilde{\tau}([0,p))$.

Since $b \in (L \setminus Z) \setminus \Sigma$, $\alpha \in \mathfrak{N}_b$. Thus there exists $G \in \mathcal{R}_b$ such that $\operatorname{mon} G = (y-b)^{\alpha}$ and $G - (y-b)^{\alpha} \in \widehat{\mathcal{O}}_b^{\mathfrak{N}_b} \subset \widehat{\mathcal{O}}_b^{\mathfrak{N}}$. Here we identify $\widehat{\mathcal{O}}_b$ with $\mathbb{R}[[y-b]]$. Since $\alpha \notin \mathfrak{N}^{-}(\alpha) = \mathfrak{N}$, we get $G \in \widehat{\mathcal{O}}_b^{\mathfrak{N}}$, and therefore $D^{\beta}G = 0$ for $\beta \in \mathfrak{N}$. Of course $J_b^l G \in J^l(b) \otimes_{\mathbb{R}} \widehat{\mathcal{O}}_b$.

Let $\underline{a} = \varepsilon(0)$. The mapping $\underline{\varphi} : M_{\varphi}^{q} \to \mathbb{R}^n$ induces a homomorphism $\widehat{\underline{\varphi}}_{\underline{a}}^{*} : \widehat{\mathcal{O}}_b \to \widehat{\mathcal{O}}_{M_{\varphi}^{q},\underline{a}}$, and further the homomorphism

$$(1) \quad J_{\underline{a}}^l \varphi : J^l(b) \otimes \widehat{O}_{M_{\varphi}^{q},\underline{a}} \to \bigoplus_{i=1}^{q} J^l(a_i) \otimes \widehat{O}_{M_{\varphi}^{q},\underline{a}}.$$

Let $\widehat{\xi}_{\underline{a}} := (J_b^l G)\circ\widehat{\underline{\varphi}}_{\underline{a}}$. Thus $\widehat{\xi}_{\underline{a}} \in J^l(b)^{\mathfrak{N}} \otimes \widehat{O}_{M_{\varphi}^{q},\underline{a}}$ and we can write $\widehat{\xi}_{\underline{a}} = (\widehat{\eta}_{\underline{a}}, \widehat{\zeta}_{\underline{a}})$ according to the direct decomposition

$$J^l(b)^{\mathfrak{N}(\alpha)} = J^l(b)^{\mathfrak{N}^{-}(\alpha)+} \oplus (\widehat{m_y}^{>\alpha} \cap J^l(b)^{\mathfrak{N}(\alpha)}).$$

Therefore the α^{th} component of $\widehat{\eta}_{\underline{a}}$ is 1 (since $D^{\alpha}G = 1$).

The restriction of $J_{\underline{a}}^l\varphi$ to $J^l(b)^{\mathfrak{N}} \otimes \widehat{O}_{M_{\varphi}^{q},\underline{a}}$ can be interpreted as a matrix $\Phi_{\underline{a}} = (\underline{A}, \underline{B})$ with entries in $\mathcal{O}_{M_{\varphi}^{q},\underline{a}} \subset \widehat{\mathcal{O}}_{M_{\varphi}^{q},\underline{a}}$. Thus we get $\Phi(\underline{a}) = (\underline{A}(\underline{a}), \underline{B}(\underline{a}))$.

Remark 4.2 The entries of matrix $\Phi_{\underline{a}}$ are globally Q-analytic functions on M_{φ}^{q}. We consider matrix $\Phi_{\underline{a}}$ locally at point $\underline{a} \in M_{\varphi}^{q}$.

Observation. If $G \in \mathcal{R}_{\underline{a}}$, we have

$$J_{\underline{a}}^l\varphi(J_b^l G) = 0.$$

It is an immediate consequence of the following formula

$$(2) \quad J_{a}^l\varphi((\widehat{\varphi}_a^{*}(D^{\beta}G))_{|\beta|\le l}) = (D^{\alpha}(\widehat{\varphi}_a^{*}(G)))_{|\alpha|\le l},$$

for all a_i such that $\underline{a} = (a_1, \ldots, a_q)$.

Hence $\widehat{\xi}_{\underline{a}}$ is a solution to the system of linear equations with Q-analytic coefficients at $\underline{a} \in M_\varphi^q$ that corresponds to the matrix $\Phi_{\underline{a}}$:

$$(\star) \quad \Phi_{\underline{a}} \cdot \widehat{\xi}_{\underline{a}}.$$

In order to prove Proposition 4.1, we introduce the concept of essential point.

Definition 4.2. A point $\underline{a} \in M_\varphi^q$ is called the essential point if the following implication is true: if $\xi = (\eta, \zeta) \in \operatorname{Ker} \Phi(\underline{a})$, then $\xi \in \operatorname{Ker}(A(a'), B(a'))$ for all $a' \in \varphi^{-1}(\underline{\varphi}(\underline{a}))$ (here $A(a')$ and $B(a')$ are matrices as in Chapter 3).

We have the following

Lemma 4.1. *The set $S \subset M_\varphi^q$ of essential points is a quasi-subanalytic set and $\underline{\varphi}(S) = X \setminus Z$.*

Proof. Clarly $S = \{\underline{a} \in M_\varphi^q : \forall_{\underline{a}'} \in M_\varphi^q \operatorname{rank} \Phi(\underline{a}) \geq \operatorname{rank} \Phi(\underline{a}')\}$ thus it is quasi-subanalytic set. Finally, $\underline{\varphi}(S) = X \setminus Z$, by Lemma 3.3 for $p = q = \binom{n+l}{l}$. □

We shall consider the numerical system of linear equations obtained form $(\star)$ by evaluating its coefficients at point $\underline{a}'$ of arc $\tilde{L}$ near $\underline{a}$. Our goal is to find solutions $\xi(\underline{a}')$, $\underline{a}' \in \tilde{L}$, whose α-th component $\xi^\alpha(\underline{a}') \neq 0$.

To this end, consider the pull-back of system $(\star)$ by mean of the parametrization $\varepsilon(t)$ of the arc $\tilde{L}$:

$$\Phi_0 \cdot \widehat{\xi}_0 = 0, \text{ with } \Phi_0 = \widehat{\varepsilon}_0^*(\Phi_{\underline{a}}),\ \widehat{\xi}_0 = \widehat{\varepsilon}_0^*(\widehat{\xi}_{\underline{a}}) \in J^l(b) \otimes \widehat{Q}_1.$$

The coefficients of the system of linear equations obtained belong to the quasianalytic local ring (Q_1, m), which is a discrete valuation ring, and therefore a noetherian ring with good algebraic properties. Hence and by Lemma 1.10, there exists a solution $\xi \in J^l(b)^{\mathfrak{N}} \otimes Q_1$ such that $\widehat{\xi} - \xi \in J^l(b) \otimes \widehat{m}$. Therefore, since $\widehat{\xi}^\alpha(0) = 1$, we get $\xi^\alpha(0) = 1$ and $\xi^\alpha(t) \neq 0$ for t close to 0.

In this manner, we achived numerical solutions $\xi(\varepsilon(t)) := \xi(t)$ of the system $(\star)$ at points $\underline{a}' = \varepsilon(t)$ lying on the arc $\tilde{L}$ near $\underline{a}$. At this stage, we are going to complete the proof.

Take a polynomial f on $\mathbb{R}^n$ such that $J^l f(b') = \xi(\underline{a}')$, $\underline{\varphi}(\underline{a}') = b'$. Then $\varphi_{\underline{a}'}^*(f) \in m_{\underline{a}'}^{l+1}$ for all $a' \in \varphi^{-1}(b')$. Hence and by the uniform Chevalley estimate $(l_{\varphi^*}(b', k) \leq l_{\varphi^*}(k) = l)$, there exists $g \in \mathcal{R}_{b'}$ such that $f - g \in \widehat{m}_{b'}^{k+1}$. Then $J^l g(b') = (\eta(\underline{a}'), \zeta)$ with some component ζ, and $\eta^\alpha(\underline{a}') \neq 0$. Since $g \in \mathcal{R}_{b'}$ and $\mathfrak{N}_{b'}^-(\alpha) = \mathfrak{N}$, $\eta^\alpha(\underline{a}')$ is the only nonzero component of $\eta(\underline{a}')$. Therefore, $\exp g = \alpha$ and $\alpha \in \mathfrak{N}_{b'}$. Consequently, $(L \setminus Z) \setminus \Sigma$ is open in $L \setminus Z$, whence $Z \cup \Sigma$ is a closed subset, as asserted in Proposition 4.1 □

This completes the proof Theorem 4.2. □

Chapter 5

The Hilbert-Samuel function.

Here we provide a proof of the implication (3) $\Rightarrow$ (5), which is based on ideas similar to those from our proof of the implication (2) $\Rightarrow$ (4). Let us emphasize that Bierstone–Milman's proof from [5] does not work in the quasianalytic settings, since they use the fact that subanalytic arcs are analytic curves and their local analytic rings are noetherian. As we mentioned in Chapter 4, it is not true in the quasianalytic settings.

Let $Z \subset X$ be closed quasi-subanalytic subsets of $\mathbb{R}^n$. For $b \in X$, $\mathfrak{N}_b$ is the diagram of initial exponents of the ideal $\mathcal{R}_b = \mathcal{F}_b(X)$, and H_b denote the Hilbert-Samuel function of $\widehat{\mathcal{O}}_b/\mathcal{R}_b$:

$$H_b(k) = \dim_{\mathbb{R}} \widehat{\mathcal{O}}_b/(\mathcal{R}_b + \widehat{m}_b^{k+1}),\ k \in \mathbb{N}.$$

The set of Hilbert-Samuel functions is equiped with the standard partial ordering, i.e. for two such functions H and H', $H \leq H'$ if $H(k) \leq H'(k)$ for all $k \in N$. With respect to the ordering above, we have the following

Theorem 5.1 *Assume that X admits a quasi-subanalytic stratification such that Z is the sum of strata an the diagram $\mathfrak{N}_b$ is constant on each stratum disjoint with Z. Then H_b is Zariski-semicontinuous relatively to Z.*

Proof. As in the proof of Theorem 4.2, we can assume that X is compact. Let $b \in X$ and let $\mathfrak{N}_b$ be the diagram of initial exponents of $\mathcal{F}_b(X)$. By Corollary 2.2, $H_b(k) = \sharp\{\gamma \in \mathbb{N}^n \setminus \mathfrak{N}_b : |\gamma| \leq k\}$. It follows from the stratification by the diagram of initial exponents that the function $b \to H_b$ is constant on each stratum. Let W be a stratum disjoint with Z such that the diagram of initial exponents is constant on W and let H_W be the Hilbert-Samuel function on W. It is sufficient to prove that for each quasi-subanalytic arc L, whose interior is contained in W and the end b of L belongs to $(\overline{W} \setminus W)$, $H_b(k) \geq H_W(k)$ for each $k \in \mathbb{N}$.

Let q be the maximal number of connected components of $\varphi^{-1}(b)$ for $b \in X$. Let us consider the mapping $\underline{\varphi} : M_\varphi^q \to X$. By Corollary 1.4, there is a stratification of $M_\varphi^q = \bigcup_i M_i^q$ such that $\mathfrak{N}_{\underline{a}} = \mathfrak{N}(\mathcal{R}_{\underline{a}})$ is constant on each M_i^q, and this stratification is compatible with stratification by the diagram of initial exponents. For each $\underline{a} \in M_\varphi^s$ such that $\underline{\varphi}(\underline{a}) = b$ there is $\mathcal{R}_b \subset \mathcal{R}_{\underline{a}}$, and thus $H_b(k) \geq H_{\underline{a}}(k)$. By Proposition 3P4, there is a uniform Chevalley estimate $l(k)$ on a stratum M_i^q such that $l_{\varphi^*}(\underline{a}, k) \leq l(k)$. Whence it is sufficient to prove the following

Lemma 5.1 *Let $\tilde{L}$ be a quasi-subanalytic arc in M_φ^q and let $\underline{a}$ be the one of its ends. Suppose that $H_{\underline{x}}$ is constant on $\tilde{L} \setminus \{\underline{a}\}$ and $l_{\varphi^*}(\underline{a}, k) \leq l(k)$. Then $H_{\underline{x}}(k) \leq H_{\underline{a}}(k)$.*

Proof. Let $\varepsilon : (-1,1) \to M^q$ be a quasianalytic parametrization of $\tilde{L}$. Let $k \in \mathbb{N}$ and $l = l(k)$. Let $v_1, \ldots, v_s$ be a basis of the vector space

$$\left(\mathcal{R}_{\underline{a}} + \widehat{m}^{k+1}_{\underline{\varphi}(\underline{a})}\right) / \widehat{m}^{k+1}_{\underline{\varphi}(\underline{a})}.$$

Let $G_j \in \mathcal{R}_{\underline{a}}$ be a representation of $v_1, \ldots, v_s$. By ξ_j^l we denote the elements of $J^l(\underline{\varphi}(\underline{a})) \otimes \widehat{O}_{M^q_\varphi,\underline{a}}$ such that $\xi_j = (J_b^l G) \circ \underline{\widehat{\varphi}}_{\underline{a}}$ for $j = 1, \ldots, s$. By the notation from Chapter 2, $\xi_j^l = (\xi_j^k, \xi_j^{lk})$. For each $\underline{x} \in \tilde{L}$, we write $J^l\varphi(\underline{x})$ as a block matrix $J^l\varphi(\underline{x}) = (S^{lk}(\underline{x}), T^{lk}(\underline{x}))$. Clearly, we can assume that the rank of $T^{lk}(\underline{x})$ is constant on $\tilde{L} \setminus \{\underline{a}\}$. Put $r := \operatorname{rank} T^{lk}(\underline{x})$. By the Observation from Chapter 4, we have

$$J^l_{\underline{a}}\varphi \cdot \xi_j^l = 0, \ \ j = 1, \ldots, s,$$

where $J^l_{\underline{a}}\varphi = (S^{lk}_{\underline{a}}, T^{lk}_{\underline{a}})$ is a matrix with coefficients from $\widehat{\mathcal{O}}_{M^q_\varphi,\underline{a}}$. Then

$$(\star) \quad \operatorname{ad}^r T^{lk}_{\underline{a}} \cdot S^{lk}_{\underline{a}} \cdot \xi_j^k = 0, \ \ j = 1, \ldots, s.$$

Similarly to the proof of Proposition 4.1, we consider the pull-back of the system $(\star)$ by mean of parametrization $\varepsilon(t)$, and thus we obtain the system of linear equations with coefficients from Q_1:

$$\operatorname{ad}^r T^{lk}_0 \cdot S^{lk}_0 \cdot \rho_j^k = 0, \ \ j = 1, \ldots, s,$$

where $T_0 = \widehat{\varepsilon}_0^*(T_{\underline{a}})$, $S_0 = \widehat{\varepsilon}_0^*(S_{\underline{a}})$ and $\rho_j^k = T_0 = \widehat{\varepsilon}_0^*(\xi_j^k)$ for $j = 1, \ldots, s$.

Let $w_1, \ldots, w_p$ be the system of generators of $\operatorname{Ker} \operatorname{ad}^r T_0^{lk} \cdot S_0^{lk}$. Since $\xi_j^k(0) = v_j$ are linearly independent, $w_1, \ldots, w_p$ span a vector space of dimension $\geq s$. Since Q_1 is noetherian, there exists a system of generators $w_1(t), \ldots, w_p(t)$ of $\operatorname{Ker} \operatorname{ad}^r T_t^{lk} \cdot S_t^{lk}$, for t from the neighborhood of 0 in $[0,1)$ and

$$\operatorname{ad}^r T^{lk}(t) \cdot S^{lk}(t) \cdot w_j(t) = 0, \ \ j = 1, \ldots, p.$$

Thus, for $\underline{x} = \varepsilon(t)$ near $\underline{a}$, there exist $w_1(\underline{x}), \ldots, w_p(\underline{x})$ which span the linear space of dimension $\geq s$ and

$$\operatorname{ad}^r T^{lk}(\underline{x}) \cdot S^{lk}(\underline{x}) \cdot w_j(\underline{x}) = 0, \ \ j = 1, \ldots, p.$$

By Lemma 2.3, $\dim(\mathcal{R}_{\underline{x}} + \widehat{m}^{k+1}_{\underline{\varphi}(\underline{x})})/\widehat{m}^{k+1}_{\underline{\varphi}(\underline{x})} = \dim \operatorname{Ker} \operatorname{ad}^r T^{lk}(\underline{x}) \cdot S^{lk}(\underline{x})$. Thus for each $\underline{x} \in \tilde{L} \setminus \{\underline{a}\}$ we obtain $\dim(\mathcal{R}_{\underline{x}} + \widehat{m}^{k+1}_{\underline{\varphi}(\underline{x})})/\widehat{m}^{k+1}_{\underline{\varphi}(\underline{x})} \geq s$, whence $H_{\underline{x}} \leq H_{\underline{a}}$. This ends the proof. □

□

Chapter 6

Proof of implication (6) $\Rightarrow$ (2).

In this chapter we prove that formal semicoherence implies a stratification by the diagram of initial exponents. However the original proof from [5] and [2] can be adapted to quasi-subanalytic settings, we present a different proof. Our reasoning is based on explicit description of membership of multi-index to the diagram of initial exponents. We obtain it by an analysis of a solution of the finite system of linear equations whose coefficients are Q-analytic functions.

We need the following auxiliary lemma.

Lemma 6.1. *Let $\{\mathfrak{N}_k\}_{k\in\mathbb{N}}$ be a sequence of diagrams such that if $\mathcal{B}_k$ is a set of vertices of $\mathfrak{N}_k$, then $\mathcal{B}_{k+1} = \mathcal{B}_k \cup \{\alpha\}$, for some $\alpha \notin \mathfrak{N}_k$. Then the sequence $\{\mathfrak{N}_k\}_{k\in\mathbb{N}}$ stabilizes while k tend to infinity.*

Proof. Consider a set $\mathfrak{N} := \bigcup_{k\in\mathbb{N}} \mathcal{B}_k + \mathbb{N}^n$. Since $\mathfrak{N} + \mathbb{N}^n = \mathfrak{N}$, there exists a finite set of vertices of $\mathfrak{N}$, say $\mathcal{B}$. Let $\beta \in \mathcal{B}_k$. It is clear that $\beta \in \mathfrak{N}$. By the definition of our sequence, β must be a vertex of $\mathfrak{N}$, because it can not be generated by any element smaller than β. Thus, for any k, $\mathcal{B}_k \subset \mathcal{B}$. Since $\mathcal{B}$ is finite, the sequence stabilizes. $\square$

Let X be a closed quasi-subanalytic subset of $\mathbb{R}^n$, semicoherent relatively to closed subset $Z \subset X$. Let $X \setminus Z = \bigcup X_i$, where X_i is a quasi-subanalytic leaf from the definition of the formal semicoherence. Let $x \in \overline{X_i}$. There exists an open neighborhood V of x in $\mathbb{R}^n$ such that for any $b \in X_i \cap V$ the ideal $\mathcal{F}_b(X)$ is generated by the power series

$$(\star) \;\; f_{ij}(Y,b) = \sum_{\alpha\in\mathbb{N}^n} f_{ij,\alpha}(b)(Y-b)^\alpha,$$

where $f_{ij,\alpha}$ are Q-analytic functions which are quasi-subanalytic. Let $\alpha \in \mathbb{N}^n$ and let $\mathfrak{N}_b$ be a diagram of $\mathcal{F}_b(X)$. Then $\alpha \in \mathfrak{N}_b$ if and only if there exist $l_1, \ldots, l_r \in \mathbb{R}[[Y-b]]$ such that $\exp(l_1 f_{i1} + \cdots + l_r f_{ir}) = \alpha$. We write

$$l_k = \sum_{\beta\in\mathbb{N}^n} a_{k\beta}(Y-b)^\beta,$$

and thus

$$l_k f_{ik} = \sum_{\gamma} \left(\sum_{\theta+\beta=\gamma} a_{k\beta} f_{ik,\theta}(b) \right) (Y-b)^\gamma.$$

Consequently $H := l_1 f_{i1} + \cdots + l_r f_{ir}$ is of the form

$$H = \sum_{\gamma} \left(\sum_{t=1}^{r} \left(\sum_{\theta+\beta=\gamma} a_{t\beta} f_{it,\theta}(b) \right) \right) (Y-b)^{\gamma}.$$

Therefore $\alpha \in \mathfrak{N}_b$ if and only if there is a solution of the linear system

$$\begin{cases} \sum_{t=1}^{r} (\sum_{\theta+\beta=\gamma} a_{tb} f_{it,\theta}(b)) = 0, \ \gamma < \alpha \\ \sum_{t=1}^{r} (\sum_{\theta+\beta=\gamma} a_{tb} f_{it,\theta}(b)) = 1, \ \gamma = \alpha \end{cases}.$$

This linear system can be expressed as a matrix with coefficients $f_{it,\theta}(b)$ of the following form

$$\begin{bmatrix} A_1 & 1 \\ A_2 & 0 \\ \vdots & \vdots \\ A_s & 0 \end{bmatrix},$$

where $s = |\alpha|$ and A_j are appropriate lines. Let α^+ be the smallest element of $\mathbb{N}^n$, which is larger than α. Therefore a matrix of linear system, which decides wether $\alpha^+ \in \mathfrak{N}_b$ or not is of the following form

$$\begin{bmatrix} B & B_1 & 1 \\ A_1 & 0 & 0 \\ \vdots & \vdots & \\ A_s & 0 & 0 \end{bmatrix},$$

where B and B_1 are single lines, and 0 under B_1 are blocks of zeros. Let us notice that this system has a solution if and only if B cannot be generated by $(A_i)_{i \in \{s,\ldots,1\}}$. Now we are ready to prove the following theorem

Theorem 6.1. *Let $Z \subset X$ be closed quasi-subanalytic subsets of $\mathbb{R}^n$ such that X is semicoherent relatively to Z. Then there exists a stratification of X into quasi-subanalytic leaves such that the diagram of initial exponents is constant on each stratum disjoint with Z and Z is a sum of strata.*

Proof. Let $X = \bigcup X_i \cup Z$, where X_i are strata disjoint with Z. It is enough to prove theorem for closure of leaves. We proceed by the induction on $k = \dim X_i$. If $k = 0$ then X_i is a single point so theorem is trivial. Suppose that we have theorem for $k > 0$. Let X_i be the leaf, $\dim X_i = k+1$. Let $x \in \overline{X_i}$, V—an open neighborhood of x in $\mathbb{R}^n$, relatively compact, such that for each $b \in X_i \cap V$ the ideal $\mathcal{F}_b(X)$ is generated by the power series as in $(\star)$. We define inductively a sequence of sets:

$$V_{(0,\ldots,1)} := \begin{cases} S = \{b \in V \cap X_i : \ (0,\ldots,1) \in \mathfrak{N}_b\}, if\ S \neq \emptyset \\ V \cap X_i, \ otherwise. \end{cases}$$

$$V_{\alpha^+} := V_\alpha \setminus \{b \in V_\alpha : \ \alpha^+ \notin \mathfrak{N}_b\}.$$

It is clear that $V_{(0,\dots,1)}$ is an open set in $V \cap X_i$, since $(0,\dots,1) \notin \mathfrak{N}_b$ if and only if $b \in \bigcup f_k^{-1}(0)$, where f_k are Q-analytic functions from the single line A_1, and thus $\{b : (0,\dots,1) \notin \mathfrak{N}_b\}$ is closed Q-analytic set contained in $V \cap X_i$. By the induction on α, V_{α^+} is an open and dense set in V_α. Since the matrix A has a constant rank on V_α, the set $\{b : \alpha^+ \notin \mathfrak{N}_b\}$ is a closed, nowhere dense Q-analytic set in V_α.

Let $\mathfrak{N}_b(\alpha)$ be a diagram generated by all elements of $\mathfrak{N}_b$ less or equal to α for $b \in V_\alpha$. It is clear that $\mathfrak{N}_b(\alpha)$ is constant on V_α. Let us notice that $V_{\alpha^+} \neq V_\alpha$ if and only if α^+ is a vertex of $\mathfrak{N}_b$ for $b \in V_{\alpha^+}$. Thus we obtain a sequence $\{\mathfrak{N}_b(\alpha)\}_{\alpha\in\mathbb{N}^n}$. By Lemma 6.1, $\mathfrak{N}_b(\alpha)$ stabilizes. Therefore V_α also stabilizes. Moreover, $\widehat{V} := \bigcap_\alpha V_\alpha$ is a finite intersection and thus an open quasi-subanalytic set, dense in $V \cap X_i$. Also the diagram of initial exponents is constant on $\widehat{V}$.

By the induction assumption for the set $(V \cap X_i) \setminus \widehat{V}$, which is a Q-analytic set, closed in $V \cap X_i$, $(V \cap X_i) \setminus \widehat{V}$ can be stratified into finitely many quasi-subanalytic leaves for which the diagram of initial exponents is constant. Each X_i can be covered by the open sets V and this cover is locally finite. Thus we obtain new stratification of X such that the diagram is constant on each stratum. By possible partition of Z, we can obtain a stratification compatible with Z. □

Chapter 7

The remaining implications.

In this chapter we discuss the proof of the remaining implications which are necessary to complete the proof of Theorem 0.1. All proofs of this implications are almost verbatim adaptations of proofs from [5]. Thus we do not present all reasonings but we indicate the crucial moments which decides why the proofs by E. Biesrstone and P. Milman can be applied in quasianalytic settings.

(3) $\Rightarrow$ (1) **and** (1) $\Rightarrow$ (2)**.** Here we present a path of reasoning of two implications: the stratification by the diagram of initial exponents implies a composite function property and a composite function property implies the uniform Chevalley estimate. First of all we recall some facts about Whintey fields.

Let N be a $\mathcal{C}^\infty$ manifold. Let $b \in N$ and $(y_1, \ldots, y_n)$ be a local coordinate chart of N at b. Then we identify a local ring $\widehat{\mathcal{O}}_b$ with a ring of formal power series $\mathbb{R}[[y-b]]$. For $g \in \mathcal{C}^\infty(N)$, we denote the Taylor series of g at b by $\widehat{g}_b$, and

$$\widehat{g}_b = \sum_{\beta \in \mathbb{N}} \frac{1}{\beta!} \frac{\partial^{|\beta|} g(b)}{\partial y^\beta} (y-b)^\beta.$$

Let M be a $\mathcal{C}^\infty$ manifold and let $\varphi : M \to \mathbb{R}^n$ be a $\mathcal{C}^\infty$ mapping, $\varphi = (\varphi_1, \ldots, \varphi_n)$. Then, for a local coordinates $(x_1, \ldots, x_m)$, we write

$$\widehat{\varphi}_a = (\widehat{\varphi}_{1,a}(x), \ldots, \widehat{\varphi}_{n,a}(x))$$

for a vector of the Taylor series of φ in a neighborhood of a in M.

Definition 7.1. Let X be a locally closed set in $\mathbb{R}^n$. Let G be a power series

$$G(b, y) = \sum_{\beta \in \mathbb{N}} \frac{G_\beta(b)}{\beta!} (y-b)^\beta.$$

We call G a $\mathcal{C}^\infty$ Whitney field on X if G is a field of Taylor series of some smooth function defined in a neighborhood of X. In other words, there exist a smooth function g, such that for each $b \in X$, $G(b, y) = \widehat{g}_b$. We recall the following three lemmas:

Lemma 7.1.([5], Lemma 11.2) *Let T be a $\mathcal{C}^k$ submanifold of $\mathbb{R}^n$, $k \in \mathbb{N} \setminus \{0\}$. Let*

$$G(b,y) = \sum_{\beta \in \mathbb{N}} \frac{G_\beta(b)}{\beta!}(y-b)^\beta,$$

where $b \in T$, and each $G_\beta \in \mathcal{C}^k(T)$. Let $\varphi : M \to \mathbb{R}^n$ be a smooth mapping, let Γ be a $\mathcal{C}^k$ submanifold of M such that $\varphi(\Gamma) \subset T$. Let $F(a,x)$ be a field of formal power series on Γ given by the formula

$$F(a,x) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} F_\alpha (x-a)^\alpha = G(\varphi(a), \widehat{\varphi}_a(x)).$$

Then each $F_\alpha \in \mathcal{C}^k(\Gamma)$ and for all $a \in \Gamma$ and vector $u \in T_a\Gamma$, we have

$$D_{a,u}F(a,x) - D_{x,u}F(a,x) = D_{b,v}G(\varphi(a), \widehat{\varphi}_a(x)) - D_{y,v}G(\varphi(a), \widehat{\varphi}_a(x)),$$

where v is an image of u by the derivative of φ and $D_{a,u}$ and $D_{x,u}$ are directional derivatives at a and x respectively in the direction u.

Lemma 7.2 ([7], Proposition 3.2). *Let T be a smooth submanifold of $\mathbb{R}^n$ and let*

$$G(b,y) = \sum_{\beta \in \mathbb{N}^n} \frac{G_\beta(b)}{\beta!}(y-b)^\beta$$

be a field of formal power series on T. Then G is a $\mathcal{C}^\infty$ Whitney field on T if and only if each $G_\beta \in \mathcal{C}^1$ and $D_{b,v}G(b,y) = D_{y,v}G(b,y)$ for every $b \in T$ and vector v from tangent space T_bT.

Lemma 7.3.([7])(Hestenes's Lemma). *Let $B \subset A$ be closed quasi-subanalytic subsetsets of $\mathbb{R}^n$. Let*

$$G(b,y) = \sum_{\beta \in \mathbb{N}^n} \frac{G_\beta(b)}{\beta!}(y-b)^\beta$$

be a field of formal power series on A, $G_\beta \in \mathcal{C}^0(A)$, such that G restricts to a $\mathcal{C}^\infty$ Whitney field on $A \setminus B$ and B. Then G is a $\mathcal{C}^\infty$ Whitney field on A.

The proof of Hestenes's lemma in [7] is presented for r-regular set. We say that a compact set A is r-regular if there exists constant $C > 0$ such that for each $a, b \in A$ exists a rectifable curve $\gamma \subset A$ of length $|\gamma| \leq C|a-b|^{1/r}$. Due to van den Dries and Miller ([12], 4.15. Whitney regularity) compact and connected quasi-subanalytic sets are r-regular. Therefore Hestenes's lemma holds for quasi-subanalytic sets.

Theorem 7.1. *Let $X \supset Z$ be closed quasi-subanalytic subsets of $\mathbb{R}^n$ such that X has a quasi-subanalytic stratification where Z is a sum of the strata* *and for each stratum T outside Z, the diagram $\mathfrak{N}_b = \mathfrak{N}(\mathcal{F}_b(X))$ is constant on T. Then (X, Z) has the composite function property: if $\varphi : M \to \mathbb{R}^n$ is a proper Q-analytic mapping such that $\varphi(M) = X$, then:*

$$\varphi^*\mathcal{C}^\infty(\mathbb{R}^n, Z) = (\varphi^*\mathcal{C}^\infty(\mathbb{R}^n, Z))\widehat{},$$

where $(\varphi^*\mathcal{C}^\infty(\mathbb{R}^n, Z))\widehat{} = (\varphi^*\mathcal{C}^\infty(\mathbb{R}^n))\widehat{} \cap \mathcal{C}^\infty(M, \varphi^{-1}(Z))$.

We shall prove that we can reduce a proof of Theorem 7.1 to a compact quasi-subanalytic set.

Suppose that the assertion of Theorem 7.1 holds for a compact quasi-subanalytic sets. Let $\varphi(M) = X$, where $\varphi : M \to \mathbb{R}^n$ is a proper Q-analytic mapping from Q-manifold M. Let $\{\omega_i\}_{i\in\mathbb{N}}$ be a $\mathcal{C}^\infty$ partition of unity in $\mathbb{R}^n$, $\omega_i : \mathbb{R}^n \to \mathbb{R}$, and let $\{K_i\}_{i\in\mathbb{N}}$ be a family of compact quasi-subanalytic sets such that $\overline{\operatorname{supp}(\omega_i)} \subset \operatorname{int} K_i$ for each $i \in \mathbb{N}$. We can arrange this partition in such way that $\{K_i\}$ are compact quasi-subanalytic sets. Let $f \in (\varphi^*\mathcal{C}^\infty(\mathbb{R}^n, Z))\widehat{}$. It is clear that $f = \sum_i (\omega_i \circ \varphi) \cdot f$. Since φ is a proper mapping then $\overline{\operatorname{supp}(\omega_i \circ \varphi)}$ is a compact set contained in the interior of compact quasi-subanalytic set $\varphi^{-1}(K_i)$. By the uniformization theorem, there exist Q-analytic manifold N and Q-analytic map $\Phi : N \to M$ such that $\Phi(N) = \varphi^{-1}(K_i)$. Let $f_i = (\omega_i \circ \varphi) \cdot f$ and $\varphi \circ \Phi = \psi$. Since $f_i \in (\varphi^*\mathcal{C}^\infty(\mathbb{R}^n, Z))\widehat{}$ and by the assumption that Theorem 7.1 holds for the compact sets, $f_i \circ \Phi = g_i \circ \psi$, for some $g_i \in \mathcal{C}^\infty(\mathbb{R}^n, Z)$, and $f_i = g_i \circ \psi$ on $\Phi(N)$. Let

$$\kappa_i(x) = \begin{cases} 1, \ x \in \operatorname{supp}(\omega_i) \\ 0, \ x \in \mathbb{R}^n \setminus K_i \end{cases}$$

be a $\mathcal{C}^\infty$ function. Then $f_i = (g_i \circ \varphi) \cdot (\kappa_i \circ \varphi) = (g_i \cdot \kappa_i) \circ \varphi$, and

$$f = \sum_i f_i = \sum_i (g_i \cdot \kappa_i) \circ \varphi.$$

Therefore $g = \sum_i g_i \cdot \kappa_i$ is a $\mathcal{C}^\infty$ function such that $f = g \circ \varphi$ and thus $f \in (\varphi^*\mathcal{C}^\infty(\mathbb{R}^n, Z))\widehat{}$.

Since we have reduced the problem to a quasi-subanalytic compact sets, we may assume that X is compact. By [5], to prove Theorem 7.1 it is sufficient to prove the following

Proposition 7.1.([5], Proposition 11.6.) *Let $A \subset X$ be a closed quasi-subanalytic subset of $\mathbb{R}^n$ of dimension d. Then there exists a subset $A' \subset A$ such that:*
(1) A' is quasi-subanalytic of dimension $< d$,
(2) If $f \in \mathcal{C}(M)^\infty$ is flat on $\varphi^{-1}(A' \cup Z)$ and $f \in (\varphi^*\mathcal{C}^\infty(\mathbb{R}^n))\widehat{}$, then there exists $g \in \mathcal{C}^\infty(\mathbb{R}^n, Z)$ such that $f - g \circ \varphi$ is flat on $\varphi^{-1}(A)$.

For the proof of Proposition 7.1, we need two following lemmas:

Lemma 7.4([5], Lemma 11.7). *For all $\gamma \in \mathbb{N}^n$, $G_\gamma \in \mathcal{C}^\infty(T)$ and $\lim_{b\to\overline{T}\setminus T} G_\gamma = 0$.*

Lemma 7.5([5], Lemma 11.8). *$G(b, y)$ is a $\mathcal{C}^\infty$ Whitney field on T.*

The implication (3) $\Rightarrow$ (1) is a consequence of Proposition 7.1, which proof is based on Lemma 7.1, Lemma 7.2, Hestenes's lemma (Lemma 7.3) and Lemma 7.4. All this lemmas hold in quasi-subanalytic settings. The crucial point is a possibility of carrying over the proof of Lemma 7.4. In [5] authors used Łojasiewicz division theorem for principal ideals of analytic functions. However the general Malgrange's theorem does not hold for Q-analytic functions, a Łojasiewicz version for principal ideals is true in the Q-analytic case([23], [3]).

The proof of the implication (1) $\Rightarrow$ (2) is a straightforward adaptation of the proof of Theorem 11.9 from [5], thus we just recall the contents of this theorem without the proof.

Let $Z \subset X$ be closed quasi-subanalytic sets in $\mathbb{R}^n$. Let $||\cdot||_k^K$ be the seminorm on $\varphi^*\mathcal{C}^\infty(\mathbb{R}^n; Z)/\mathrm{Ker}\varphi^*$ (see [5], page 774). For every compact $K \subset X$ and $k \in \mathbb{N}$, there exists $l(K, k) \in \mathbb{N}$ such that $||g||_k^K \leq \mathrm{const}||\varphi^*(g)||_l^{\varphi^{-1}(K)}$ for each $g \in \mathcal{C}^\infty(\mathbb{R}^n; Z)$.

Theorem 7.2([5], Theorem 11.9) *Suppose that (X, Z) has the composite function property. Let $\varphi : M \to \mathbb{R}^n$ be a proper quasianalytic mapping such that $\varphi(M) = X$. Then*

$$l_{\varphi^*}(b, k) \leq l(K, k),$$

for all $b \in (X \setminus Z) \cap K$.

Stratification by the diagram of initial exponents implies formal semicoherence. Here we discuss the implication (3) $\Rightarrow$ (6).

Let $Z \subset X$ be closed quasi-subanalytic subsets of $\mathbb{R}^n$ such that there is a stratification of X, where Z is the sum of strata and the diagram of initial exponents is constant on each stratum outside Z. Let $\varphi : M \to \mathbb{R}^n$ be a proper Q-analytic mapping such that $X = \varphi(M)$. By Corollary 1.5, we can assume that the number of connected component of the fibre $\varphi^{-1}(b)$ is constant on each stratum $Y \subset X \setminus Z$.

Take the stratum $Y \subset X \setminus Z$, where the number of connected components of the fibre is s. Let $\underline{M}^s_\varphi$ be the same as in Chapter 3. Put $\underline{L}^s_\varphi = \underline{M}^s_\varphi \cap \underline{\varphi}^{-1}(Y)$. Then $\underline{\varphi}(\underline{L}^s_\varphi) = Y$ and $\mathcal{R}_{\underline{a}} = \mathcal{R}_{\underline{\varphi}(\underline{a})}$ for all $\underline{a} \in \underline{L}^s_\varphi$. To prove the formal semicoherence it is sufficient to repeat the proof of Proposition 9.3 from [5] in quasianalytic version. We have the following

Proposition 7.2 ([5], Proposition 9.3). *Let L denote a quasi-subanalytic leaf in M^s_φ such that $\mathfrak{N}_{\underline{a}} = \mathfrak{N}(\mathcal{R}_{\underline{a}})$ is constant on L, $\underline{a} \in L$. Let α_j, $j = 1, \ldots, t$, denote the vertices of $\mathfrak{N}_{\underline{a}}$ for each $\underline{a} \in L$. Let*

$$G^j(y) = (y - \underline{\varphi}(\underline{a}))^{\alpha_j} - \sum_{\gamma \in \mathbb{N}^n \setminus \mathfrak{N}} r^j_\gamma(\underline{a})(y - \underline{\varphi}(\underline{a}))^\gamma \;\; j = 1, \ldots, t,$$

be the standard basis of $\mathcal{R}_{\underline{a}}$. Then each r^j_γ is Q-analytic on L and quasi-subanalytic.

The proof of Proposition 7.2 is just the repetition of the proof of Proposition 9.3 from [5]. The reason why it can be carried over without any changes is the fact that the functions r_γ^j are obtained as the solutions of the system of linear equations directly from Cramer's rule, thus there are not used any tools which are forbidden in quasianalytic settings.

Semicontinuity of the Hilbert-Samuel function implies the uniform Chevalley estimate. To prove the implication (5) $\Rightarrow$ (2), and at the same time to finish the proof of Theorem 0.1, it is sufficient to prove the following

Proposition 7.3([5]). *Let $k \in \mathbb{N}$ and let L be a quasi-subanalytic leaf in M_φ^s. Suppose that $H_{\underline{a}}(k)$ is constant on L. Then there is a proper Q-analytic subset Y of L, quasi-subanalytic in M_φ^s, such that $l_\varphi^*(\underline{a}, k)$ is bounded on $L \setminus Y$.*

The proof of Lemma 5.3 is a verbatim adaptation of the proof of Lemma 10.3 by Bierstone and Milman.

Chapter 8

An example of a quasi-subanalytic semicoherent set.

In this chapter we present an example of a closed quasi-subanalytic set which is formally semicoherent and is not subanalytic set. We generalize an example of Bierstone and Milman and we show a family of semicoherent quasi-subanalytic sets. Our example rely on existence of Q-analytic functions which are nowhere analytic.

Let $I \subset \mathbb{R}$ be a closed interval and $f : I \to \mathbb{R}$ be the restriction of a Q-analytic function defined on some neighborhood of I, which is not constant. By [9] we can choose f to be nowhere analytic on the interior of I. Since f is not constant, we can assume that f is strictly increasing on I.

We need the following

Lemma 8.1. *Let Y be a quasi-subanalytic set in $\mathbb{R}^n$, which is the graph of a Q-analytic function $f : U \to \mathbb{R}$, where U is an open set in $\mathbb{R}^n$. Then, for any $b = (a, f(a)) \in Y$, the ideal $\mathcal{F}_b(Y)$ is generated by the formal power series $y - \widehat{f}_a$, where $\widehat{f}_a$ is a Taylor series of f at a.*

Proof. Let $\varphi(x) = (x, f(x))$. Then φ is a proper Q-analytic map and $\varphi(U) = Y$. Therefore $\mathcal{F}_b(Y) = \operatorname{Ker} \widehat{\varphi}_a^*$. It is clear that $\widehat{\varphi}^*(y - \widehat{f}_a) = \widehat{f}_a - \widehat{f}_a = 0$. Let $G \in \mathcal{F}_b(Y)$. To end the proof we must show that G is divisible by $y - \widehat{f}_a$. Let us consider the lexicographic ordering of $\mathbb{N}^n$ such that $(0, \dots, 1)$ is the smallest nonzero element(it corresponds to variable y). By the Grauert-Hironaka algorithm, there exist power series H and R such that $\operatorname{supp} H \subset \Delta = (0, \dots, 1) + \mathbb{N}^n$, $\operatorname{supp} R \subset \mathbb{N}^n \setminus \Delta$ and $G = H \cdot (y - \widehat{f}_a) + R$. Since $\operatorname{supp} R \subset \mathbb{N}^n \setminus \Delta$, R does not depend on y. Now $\widehat{\varphi}^*(G) = R = 0$. Therefore G is divisible by $y - \widehat{f}_a$. $\square$

Now let $I = (a, b)$ and $f : I \to \mathbb{R}$ be nowhere analytic, Q-analytic, strictly increasing function. Let us consider Q-analytic function

$$F(x) := f\left(\arctan\left(\frac{2\pi x - \pi a - \pi b}{2b - a}\right)\right).$$

F maps $\mathbb{R}$ onto $f(I)$. F is strictly increasing and nowhere analytic in $\mathbb{R}$. We can assume that

$$F(0) = 0,$$
$$F(x) < 0 \Leftrightarrow x < 0,$$
$$F(x) > 0 \Leftrightarrow x > 0.$$

Let $G(x, y, z) = z^3 - x^2 zF(y) - x^4$, and let $X := G^{-1}(0)$. X is modified set from Exapmle 1.4 from [BM-1]. It's singular locus is half-line $\{(x, y, z) : x = 0, z = 0, y \leq 0\}$. The partial derivatives of G are given by the formulas:

$$\frac{\partial G}{\partial x} = -2xzF(y) - 4x^3,$$
$$\frac{\partial G}{\partial y} = -x^2 zF'(y),$$
$$\frac{\partial G}{\partial z} = 3z^2 - x^2 F(y).$$

For $\{(x, y, z) : y < 0\}$ z can be uniquely solved as a function of x and y(see [BM-3]). For $\{(x, y, z) : x \neq 0, z \neq 0, y > 0\}$ we can uniquely solve y as a function of x and z. Therefore we can stratify X in the following way:

$$X_1 := \{(x, y, z) \in X : x > 0\, z > 0\},$$
$$X_2 := \{(x, y, z) \in X : x > 0\, z < 0\},$$
$$X_3 := \{(x, y, z) \in X : x < 0\, z > 0\},$$
$$X_4 := \{(x, y, z) \in X : x < 0\, z < 0\},$$
$$X_5 := \{(0, 0, 0)\},$$
$$X_6 := \{(x, y, z) \in X : x = z = 0\}.$$

Each X_i is a Q-analytic manifold. Let $p = (a, b, c) \in X$. By Lemma 8.1, for $p \in X_i$, where $i \in \{1, 2, 3, 4\}$, $\mathcal{F}_p(X)$ is generated by $y - \widehat{y}_{(a,c)}$. If $p \in X_6$ and $b > 0$, then $\mathcal{F}_p(X)$ is generated by $\{x - a, z - c\}$. If $p \in X_6$, and $b < 0$ thus z can be uniquely solved as function of x and y, and then $\mathcal{F}_p(X)$ is generated by $z - \widehat{z}_{a,b}$. Finally X is formally semicoherent.

We will show that X is not subanalytic. We consider subset $\{(x, y, z) \in X : y < 0\}$, where z can be uniquely solves as function of x and y. By the implicit function theorem we calculate partial derivatives of $z(x, y)$:

$$\frac{\partial z}{\partial x} = -\left(\frac{\partial G}{\partial z}\right)^{-1} \frac{\partial G}{\partial x} = \frac{2xzF(y) + 3x^4}{3z^2 - x^2 F(y)}.$$

Suppose that z is analytic. Then $\frac{\partial z}{\partial x}(x, y)$ is also analytic. We can transform the equation above to the form

$$3z^2 \cdot \frac{\partial z}{\partial x}(x, y) - x^4 = F(y)(x^2 \cdot \frac{\partial z}{\partial x}(x, y) + 2xz).$$

Last equation suggest that $F(y)$ is almost everywhere a quotient of analytic functions, which is a contradiction to assumption that F is nowhere analytic. Therefore $z(x, y)$ is also nowhere analytic and X is not a subanalytic set.

In [30] Pawłucki has given an example of a closed subanalytic set that it is not semicoherent, which shows that the properties investigated by E. Bierstone and P. Milman do not hold in general and that the sets with those properties are "tame" from the analytic point of view. The above example shows that the class of closed quasi-subanalytic sets with the properties under study is wider than that of semicoherent subanalytic sets.

Bibliography

[1] N.Bourbaki, *Elements of mathematics. Commutative algebra. Chapters 1-7.*, Springer (1998).

[2] E. Bierstone, P. Milman, *Relations among analytic functions*, I, Ann. Inst. Fourier (Grenoble) **37:1** (1987), 187-239; II,**37:2** (1987), 257-280.

[3] —, —, *Semianalytic and subanalytic sets*, Inst. Hautes Etudes Sci. Publ. Math. **67** (1988), 5-42.

[4] —, — *Geometric and differential properties of subanalytic sets*, Bull. Amer. Math. Soc. **25** (1991), 385-393.

[5] —, —, *Geometric and differential propeties of subanalytic sets*, Ann. of Math, **147** (1998), 731-785.

[6] —, —, *Resolution of singularities in Denjoy-Carleman classes*, Selecta Math. (NS) 10 (2004), 1-28.

[7] —, —, W. Pawłucki, *Composite differentiable functions*, Duke Math. J. **83** (1996), 607-620.

[8] T. Carleman, *Les fonctions quasi-analytiques*, (1926), Gauthier-Villars.

[9] R.B Darst, *Localization properties of basic classes of $\mathcal{C}^\infty$ functions*, Proc. of the Amer. Math. Soc., 46, Number 1, October 1974.

[10] A. Denjoy, *Sur les fonctions quasi-analytiques de variable réelle*, C.R. Acad. Sci. Paris 173: 1329-1331 (1921).

[11] L. van den Dries *Tame topology and o-minimal structures*, London Math. Soc. Lect. 248, Cambr. Univ. Press (1998)

[12] —, C. Miller *Geometric cathegories and o-minimal structures*, Duke Math. J. Volume 84, Number 2 (1996), 497-540.

[13] R. Hardt *Semialgebraic local-triviality in semi-algebraic mappings*, Amer. J. Math **102** (1980), 291-302.

[14] S. Łojasiewicz *Stratifications et triangulations sous-analytiques*, Geometry Seminars, 1986, Univ. Stud. Bologna, Bologna, (1986), 83-97.

[15] —, *Sur le probléme de la division*, Stuia Math. 18 (1959), 87-136, and Rozprawy Mat. 22 (1961).

[16] —, *Ensembles semi-analytiques*, Inst. Hautes Etudes Sci., Bures-sur-Yvette, 1964.

[17] B. Malgrange, *Ideals of differentiable functions*, Oxford Univ. Press, 1996.

[18] K.J. Nowak, *On two problems concerning quasianalytic Denjoy-Carleman classes*, RAAG Preprint **241** (2007) and IMUJ Preprint 2007/19.

[19] —, *On the Euler characteristic of the links of a set determined by smooth definable functions*, Ann. Polon. Math. 93 (3) (2008), 231-246.

[20] —, *Decomposition into special cubes and its application to quasi-subanalytic geometry*, Ann. Polon. Math. **96** (2009), 65-74.

[21] —, *Quantifier elimination, valuation property and preparation theorem in quasianalytic geometry via transformation to normal crossings*, Ann. Polon. Math. **96**(2009), 247-282.

[22] —, *Addendum to the paper "Decomposition into special cubes and its application to quasi-subanalytic geometry"*, Ann. Polon. Math. **98**(2010), 201-205.

[23] —, *Division of distributions by locally definable quasianalytic functions*, Bulletin of the Polish Academy of Sciences. Mathematics **58**(2010), 201-209.

[24] —, *A note on Bierstone-Milman-Pawłucki's paper "Composite differentiable functions"*, Ann. Polon. Math. **102** (2011), 293-299.

[25] —, *Supplement to the paper "Quasianalytic perturbation of multiparameter hyperbolic polynomials and symmetric matrices"*, Ann. Polon. Math. **103** (2012), 101-107

[26] —, *Quasianalytic structures revisited: Quantifier elimination, valuation property and rectilinearization of functions*, RAAG Preprint **340**(2012) and IMUJ Preprint 2012/2

[27] —, *On the singular locus of sets definable in a quasianalytic structure*, Adv. Geom. **13**(2013), 619-632.

[28] —, *On division of quasianalytic function germs*, Int. J. Math. **13** (2013). 1-5.

[29] —, *An example concerning quantifier elimination in quasianalytic structures*, arXiv:1310.1303 [math.AG] (2013), v.2.

[30] W. Pawłucki, *On relations among analytic functions and geometry of subanalytic sets*, Bull. Polish Acad. Sci Math **37** (1989), 117-125.

[31] J-P. Rolin, P. Speissegger, A.J. Wilkie, *Quasianalytic Denjoy-Carleman classes and o-minimality*, J. Amer. Math. Soc. 16 (2003), 751-777.

[32] H. Sfouli, *On a problem concerning quasianalytic local rings*, Ann. Polon. Math. 111 (2014), 13-20.

[33] V. Thilliez, *On quasianalytic local rings*, Expo. Math. **26** (2008), 1-23.

[34] J.Cl. Tougeron, *An extension of Whitney's spectral theorem*, Inst. Hautes Études Sci. Publ. Math. **40** (1971), 139-148.

[35] —, *Idéaux de Functions Différentiables*, Springer Verlag, Berlin, 1972.

[36] H. Whitney, *Differentiable even functions.*, Duke Math. J. 10 (1943), 159-160.